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On the validity of several previously published perturbation formulas for the acoustoelastic effect on Rayleigh waves

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ABSTRACT

This article revisits the evaluation by a perturbation theory of the modification of the Rayleigh wave velocity under a static loading varying with depth. Two derivations, that have been exposed in the past and presented as comparable, are questioned. A new derivation of the perturbation formula is given by adapting Auld's approach. Validation with exact calculations is provided. The examples cover depth-varying static stress as well as depth-varying third order elastic properties.

1. Introduction

The slight modification of sound wave velocities when the propagation medium is statically stressed has been extensively studied in the past [1,2]. This effect is known as the acoustoelastic effect. The possibility of using it to monitor the state of residual stresses inside a material has been widely considered, and numerous applications have been developed in fields where either unwanted tensile stresses or deliberately generated compressive stresses play a major role on the lifetime of mechanical components. Previous works covered virtually all types of waves (bulk waves as well as surface or other guided waves). Because the strains involved are small, perturbation theory has been a dominant approach to predict the magnitude of the effect.

We shall in this work focus on the Rayleigh surface wave. This field has taken benefit from other communities which were already involved in studying the influence of depth-dependent texture on the dispersive character of the Rayleigh wave. A milestone was Auld's [3] perturbation theory, which lays on reciprocity relationships under a first order Born approximation (see Szabo [4] and Tittmann et al. [5] for early examples of application). The first work to deal with depth-varying loadings was probably that of Hirao et al. [6]. These authors used a perturbation approach to derive a formula which predicts a frequency-dependent behavior in the velocity of the Rayleigh wave, also providing experimental evidence in the case of a stress growing with depth. A few years later, Husson [7] addressed the same problem by using another way to derive the perturbation formula, based on an adaptation of Auld's methodology. Ditri and Hongerholt [8] later corrected typographical errors. Both articles of Hirao et al. and of Husson are today widely cited. Still, they do not agree.

This article is organized as follows. First, arguments are given to prove that neither the formula derived by Hirao et al. nor Husson's can cover arbitrary profiles of loading, and steps in both demonstrations referring to this fact are identified. Second, a new derivation of the perturbation formula is given by adapting Auld's approach. A general formula is given, and then applied to an initially isotropic half space. Finally, the several sets of formulas are compared numerically on diverse examples, together with a validation by an exact calculation.

2. Preliminary arguments

In what follows, ε_{ij}^{S} refers to the static strain, k to the wavenumber, x_1 is the coordinate in the direction of propagation, x_3 is the vertical coordinate and the over-bar means a value at the surface ($x_3 = 0$).

The formula derived by Hirao et al. expresses the variation of velocity of the Rayleigh wave Δv_R as a linear combination of $\bar{\varepsilon}_{ii}^{S}$, $\partial_3\bar{\varepsilon}_{ii}^{S}/k$, $\partial_3^2\bar{\varepsilon}_{ii}^{S}/k^2$, and integrals of $\varepsilon_{ii}^{S}(x_3)$ over the half-space weighted by decreasing exponentials. The formula derived by Husson has some common and some different features. It expresses the variation of velocity as a linear combination of $\bar{\varepsilon}_{11}^{S}$ and integrals of $\varepsilon_{ii}^{S}(x_3)$ over the half-space weighted by decreasing exponentials. In both cases, the presence of terms that explicitly depend on the value at the surface of ε_{ij}^{S} and its first two derivatives is problematic. Indeed, if we consider a loading which is located near the surface, *i.e.* which has a finite extent in depth, then the integral terms can be shown to tend to zero at low frequencies. The predicted low frequency behavior would then be of the form $\Delta v_R^{(\mathrm{LF})} = \beta_0 + \beta_1/k + \beta_2/k^2$, which has a non-null, potentially divergent value for $k \to 0$. This is in contradiction with the physical intuition that for a localized loading the low frequency limit of the velocity should be

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only determined by the unmodified substrate. Therefore, both formulas are restricted to some cases which exclude the low frequency limits of localized profiles of loading.

The demonstration of Hirao et al. follows the strategy of first obtaining a perturbed solution to the wave equation. A wave field potential F is decomposed into a zeroth order and a first order term, labelled $F = F^0 + F^1$. The differential equation satisfied by F^1 has a homogeneous part identical to the original wave equation, and an inhomogeneous part involving the static stress field and F^0. By using the plane waves of the unperturbed medium, a particular solution for F^1 is constructed. Then, F is inserted into the boundary condition and a system is obtained whose determinant must vanish. This last step provides an explicit expression for the variation of velocity. The method, which is standard and, in principle, correct, is however truly cumbersome as it requires to, first, expand the inhomogeneous part of the (fourth order) differential equation satisfied by F^1, second, construct explicitly a particular solution by integrating this inhomogeneous part multiplied by products of the (four) linearly independent solutions, and finally insert the whole expansion into the boundary condition which involves several derivative operators. The expressions to deal with are thus growing considerably at each step, and it would be a true challenge to re-derive them to obtain an error-proof formula. Nevertheless, the following mistake can be identified. Hirao et al. wrote the inertial term in the wave equation $\mu V^2/V_T^2$ instead of ρV^2, to anticipate a further division by μ. This is of no consequence for the unperturbed equation, but apparently misled them to write the corresponding first order variation $2\Delta V\, V_0/V_{T,0}^2 \equiv z$ instead of $2\Delta V\, V_0/V_{T,0}^2 + \Delta\rho\, V_0^2/\mu = z - \varepsilon_{NN}\, V_0^2/V_{T,0}^2$. Indeed, in the R_i expression (see Eq. (35) in [6]), the factors of z should also be present multiplied by $-r_0\, V_0^2/V_{T,0}^2$ in the $L_i^{(2)}$ constants (see Eq. (36) in [6]), which is not the case. R_i is used to generate the particular solution F^1, so this error impacts the final formula, even though, as will be shown below, it seems to affect only slightly the predicted dispersion in the particular case considered by Hirao et al. We did not try to find further errors, nor to correct them, as we chose to follow a more compact method to obtain the perturbation formula.

Husson's demonstration is more attractive in the sense that it avoids dealing explicitly with most of the perturbed terms, and results in handling more compact expressions. By multiplying the perturbed and unperturbed fields and integrating them over a well-chosen volume, the variation of phase $\delta\Phi$ is expressed as an integral over the static stresses weighted by the unperturbed field. However, Husson's derivation is done using the wave equation expressed in the material coordinate system, *i.e.* coordinates which are deformed together with the material. As a consequence, the phase shift defined this way is bound to the deformation of the distances and must be transformed back into the space coordinate system before the variation of velocity can be deduced from it. This final step is presented as $\Delta v_R/v_R = (\Delta L/L) - \delta\Phi\, v_R/(\omega L)$, with $\Delta L/L = \bar{\varepsilon}_{11}^{\mathrm{S}}$. It however happens to be a special case of a more general formula, and is valid only if the strain is uniform. So, in practice, Husson's formula is limited to uniform deformations, unless the latter transformation is replaced by the general one. Notice that Husson's article adapted its methodology from an earlier work by Husson and Kino [9] on bulk waves propagating in inhomogeneously strained media. This latter article should therefore also be considered carefully. To obtain a correct formula one strategy could be to derive this corrective term. Another one could be to re-derive the perturbation formula from the wave equation expressed in the space coordinate system, in which the velocity is measured. We have done both, although we decided to present the latter one in this article because it leads to dispersion equations that can also be solved exactly using standard numerical procedures. We will devote in the near future another article to the derivation of the general form of the corrections. Meanwhile, as a hint, we give here the general form of the relation between velocity variation and phase shift expressed in the material coordinates. By adapting Husson's demonstration to the wave equation expressed in the space coordinate system, one can define a phase shift $\delta\phi$ such as $\Delta v_R/v_R = -\delta\phi\, v_R/(\omega L)$. By transforming some terms of $\delta\phi$ into the material coordinate system, one can obtain the difference between both definitions of phase shifts:

$$\delta\phi - \delta\Phi = -\frac{\omega}{P}\int_V \frac{\mathscr{K}-\mathscr{E}}{2}\partial_N u_N^{\mathrm{S}}\,\mathrm{d}V - \frac{\omega}{P}\int_V \mathscr{E}_{ij}\,\partial_j u_i^{\mathrm{S}}\,\mathrm{d}V, \tag{1}$$

in which the power flow P and the densities of kinetic energy $\mathscr{K} = \frac{1}{2}\rho_0\,|\mathbf{v}_0|^2$ and elastic energy $\mathscr{E} = \mathscr{E}_{NN}$, $\mathscr{E}_{ij} = \frac{1}{2}Re\{\partial_i \mathbf{u}_0^{*}\cdot(\boldsymbol{\sigma}_j)_0\}$ of the dynamic field in the unperturbed medium have been defined. In the case of a Rayleigh wave in an isotropic medium, one can show that only $\mathscr{E}_{11}$ and $\mathscr{E}_{33}$ are not null, with furthermore $\int_V \mathscr{E}_{11}\,\mathrm{d}V = L\,P/v_R$, $\int_V \mathscr{E}_{33}\,\mathrm{d}V = 0$, $\int_V (\mathscr{K}-\mathscr{E})\mathrm{d}V = 0$, *i.e.* $\delta\phi - \delta\Phi = -\bar{\varepsilon}_{11}^{\mathrm{S}}(\omega L/v_R)$ if the static strain is homogeneous.

3. Basic equations

Let us consider a half space which mechanical properties are invariant in the planar (x_1, x_2) directions but may vary in the vertical x_3 direction. In its natural state, *i.e.* in absence of any mechanical deformation, the medium is described by a mass density $\rho(x_3)$, a stiffness tensor $C_{ijkl}(x_3)$ and third order elastic moduli $C_{ijklmn}(x_3)$, and its surface is isolated from any other medium. At first no assumption is made on the symmetry of the medium, although isotropy will be assumed in the next sections. A static stress $\sigma_{ij}^{\mathrm{S}}(x_3) = C_{ijkl}\,\partial_k u_l^{\mathrm{S}}(x_3)$ is applied and defines the initial state. Except when specified, the coordinates and derivatives refer to this state. Then, a mechanical wave of small additional amplitude is considered and defines the final state (referred to with superscript $^{\mathrm{f}}$). We shall be interested in a wave guided by the surface and propagating in the x_1 direction.

Following Pao et al. [10], in the space coordinate system defined by the initial state, the incremental displacement u_i and the difference between the final state second Piola-Kirchhoff and initial Cauchy stress tensors $T_{ij} = T_{ij}^{\mathrm{f}} - \sigma_{ij}^{\mathrm{S}}$ are related by the wave equation and generalized Hooke's law:

$$\partial_j[T_{ij} + \sigma_{jk}^{\mathrm{S}}\,\partial_k u_i] = \rho^{\mathrm{S}}\,\partial_{t^2}^2 u_i, \tag{2}$$

$$T_{ij} = C_{ijkl}^{\mathrm{S}}\,\partial_k u_l, \tag{3}$$

with

$$\rho^{\mathrm{S}} = \rho(1 - \partial_m u_m^{\mathrm{S}}), \tag{4a}$$

$$\begin{aligned} C_{ijkl}^{\mathrm{S}} = {} & C_{ijkl}(1-\partial_m u_m^{\mathrm{S}}) + C_{ijklmn}\,\partial_m u_n^{\mathrm{S}} \\ & + C_{mjkl}\,\partial_m u_i^{\mathrm{S}} + C_{imkl}\,\partial_m u_j^{\mathrm{S}} \\ & + C_{ijml}\,\partial_m u_k^{\mathrm{S}} + C_{ijkm}\,\partial_m u_l^{\mathrm{S}}. \end{aligned} \tag{4b}$$

If ρ^{S} or C_{ijkl}^{S} are discontinuous at some depth, then the displacement and forces are continuous through this interface. This condition must be written in the final set of coordinates, in which the wave slightly additionally modifies the space. Let us refer to this incremental deformation gradient $F_{ij}^{\mathrm{f}} = \partial_j x_i^{\mathrm{f}} = \delta_{ik}(\delta_{kj} + \partial_j u_k)$ and to its determinant $J^{\mathrm{f}} = \det\mathbf{F}^{\mathrm{f}}$. The Cauchy stress (or true stress) tensor $\boldsymbol{\sigma}^{\mathrm{f}}$ is related to $\mathbf{T}^{\mathrm{f}}$ through $\boldsymbol{\sigma}^{\mathrm{f}} = (J^{\mathrm{f}})^{-1}\mathbf{F}^{\mathrm{f}}\,\mathbf{T}^{\mathrm{f}}\,{}^{\mathrm{t}}\mathbf{F}^{\mathrm{f}}$. An oriented surface element is transformed following $\mathbf{n}^{\mathrm{f}}\,\mathrm{d}s^{\mathrm{f}} = J^{\mathrm{f}}\,({}^{\mathrm{t}}\mathbf{F}^{\mathrm{f}})^{-1}\,\mathbf{n}\,\mathrm{d}s$. Using these relations, and considering $\mathbf{n} = {}^{\mathrm{t}}(001)$, the elementary force through the interface expresses as:

$$\sigma_{ij}^{\mathrm{f}}\,n_j^{\mathrm{f}}\,\mathrm{d}s^{\mathrm{f}} = \{(\sigma_{i3}^{\mathrm{S}} + T_{i3}) + (\sigma_{k3}^{\mathrm{S}} + T_{k3})\partial_k u_i\}\mathrm{d}s. \tag{5}$$

Remembering that σ_{ij}^{S} is a static stress, which therefore satisfies continuity without the presence of the incremental wave field, and neglecting the term $T_{k3}\,\partial_k u_i$ in Eq. (5), the following incremental quantity

$$\widetilde{T}_{i3} = T_{i3} + \sigma_{k3}^{\mathrm{S}}\,\partial_k u_i \tag{6}$$

is continuous through the interface. At the surface, the stress-free condition of natural state expresses as $\widetilde{T}_{i3}|_{x_3=0} = 0$.

We now suppose that the wave field is harmonic in time and in the

x_1 direction - with phase velocity v - and is invariant in the x_2 direction: $u_i = u_i(x_3)e^{i\omega(t-x_1/v)}$. Following Osetrov et al. [11], the six dimensional state vector $\boldsymbol{\Gamma} = [\widetilde{\mathbf{T}}_3, \mathrm{i}\omega\,\mathbf{u}]^t$ satisfies:

$$\partial_3\boldsymbol{\Gamma} = \mathrm{i}\omega\,\mathbf{F}(x_3, v)\boldsymbol{\Gamma}, \tag{7}$$

where the propagator $\mathbf{F}$ is

$$\mathbf{F}(x_3, v) = \frac{1}{v^2}\begin{bmatrix} v\,\mathbf{B}\,\mathbf{C}^{-1} & v^2\rho^{\mathrm{S}}\,\mathbf{I}-\mathbf{A} + \mathbf{B}\,\mathbf{C}^{-1}\mathbf{B}^t \\ v^2\,\mathbf{C}^{-1} & v\,\mathbf{C}^{-1}\mathbf{B}^t \end{bmatrix}, \tag{8}$$

$\mathbf{I}$ being the identity matrix, and

$$\mathbf{A}(x_3)=\mathbf{A}^{\mathrm{S}} + \sigma_{11}^{\mathrm{S}}\,\mathbf{I} = \begin{bmatrix} C_{11}^{\mathrm{S}} + \sigma_{11}^{\mathrm{S}} & C_{16}^{\mathrm{S}} & C_{15}^{\mathrm{S}} \\ C_{16}^{\mathrm{S}} & C_{66}^{\mathrm{S}} + \sigma_{11}^{\mathrm{S}} & C_{56}^{\mathrm{S}} \\ C_{15}^{\mathrm{S}} & C_{56}^{\mathrm{S}} & C_{55}^{\mathrm{S}} + \sigma_{11}^{\mathrm{S}} \end{bmatrix}, \tag{9}$$

$$\mathbf{B}(x_3)=\mathbf{B}^{\mathrm{S}} + \sigma_{13}^{\mathrm{S}}\,\mathbf{I} = \begin{bmatrix} C_{15}^{\mathrm{S}} + \sigma_{13}^{\mathrm{S}} & C_{14}^{\mathrm{S}} & C_{13}^{\mathrm{S}} \\ C_{56}^{\mathrm{S}} & C_{46}^{\mathrm{S}} + \sigma_{13}^{\mathrm{S}} & C_{36}^{\mathrm{S}} \\ C_{55}^{\mathrm{S}} & C_{45}^{\mathrm{S}} & C_{35}^{\mathrm{S}} + \sigma_{13}^{\mathrm{S}} \end{bmatrix}, \tag{10}$$

$$\mathbf{C}(x_3)=\mathbf{C}^{\mathrm{S}} + \sigma_{33}^{\mathrm{S}}\,\mathbf{I} = \begin{bmatrix} C_{55}^{\mathrm{S}} + \sigma_{33}^{\mathrm{S}} & C_{45}^{\mathrm{S}} & C_{35}^{\mathrm{S}} \\ C_{45}^{\mathrm{S}} & C_{44}^{\mathrm{S}} + \sigma_{33}^{\mathrm{S}} & C_{34}^{\mathrm{S}} \\ C_{35}^{\mathrm{S}} & C_{34}^{\mathrm{S}} & C_{33}^{\mathrm{S}} + \sigma_{33}^{\mathrm{S}} \end{bmatrix}. \tag{11}$$

Several numerical procedures were developed in the past to obtain the dispersion relationships $v(\omega)$ of a guided wave from Eq. (7). One of them [12], which has become classical, is based on approximating the medium as piece-wise constant and finding the roots of the determinant of the so-called "transfer matrix". We will rely on this exact method to check for the validity of the perturbation formula derived in the next section.

Auld [3] derived a perturbation theory that describes how a variation of mass density or elastic constants affects the phase velocity of such a guided wave. The formula will be recalled in the next section. At first order, the theory expresses the dispersion as an integral of the variations weighted by the unperturbed wave field. The demonstration of the formula relies on a reciprocity approach applied to a propagation equation formally similar to Eq. (8). It is to be noted that Eqs. (8)–(11) indicate that no formal difference distinguishes the effect of the σ_{22}^{S}, σ_{12}^{S} and σ_{23}^{S} components from a variation of mass density and elastic constants. This remark can even be extended to the σ_{11}^{S} component by noting that the extra variation in the diagonal of the $\mathbf{A}$ matrix can formally be transferred into an appropriately defined equivalent mass density $\rho^{\mathrm{S}}-\sigma_{11}^{\mathrm{S}}/v^2$. Therefore, in these four particular but very relevant cases, Auld's formula may be readily applied. Concerning the contribution of the remaining σ_{33}^{S} and σ_{13}^{S} components, the formal variation in the diagonals of the $\mathbf{B}$ and $\mathbf{C}$ matrices have deeper consequences and an extended formula is necessary. This will be dealt with in the next section.

4. Perturbation theory

4.1. Auld's formula

We here first recall the result of Auld's [3] perturbation theory. If we let the perturbed state be defined by $\rho \to \rho + \Delta\rho(x_3)$, $C_{ijkl} \to C_{ijkl} + \Delta C_{ijkl}(x_3)$, then the velocity of the Rayleigh wave $v_R + \Delta v_R$ deviates from its original value by:

$$\frac{\Delta v_R}{v_R}(\omega) = \frac{v_R}{4P}\int_0^\infty [-\Delta\rho\,\omega^2\,u_i^*\,u_i + \varepsilon_{ij}^*\,\Delta C_{ijkl}\,\varepsilon_{kl}]\mathrm{d}x_3, \tag{12}$$

with

$$P = \frac{\omega^2\rho\,v_R}{2}\int_0^\infty u_i^*\,u_i\,\mathrm{d}x_3 \tag{13}$$

the power flux per unit length. As explained earlier, Eq. (12) can be readily applied for the σ_{11}^{S}, σ_{22}^{S}, σ_{12}^{S} and σ_{23}^{S} components using $\Delta C_{ijkl} = C_{ijkl}^{\mathrm{S}}-C_{ijkl}$ and $\Delta\rho = \rho^{\mathrm{S}}-\rho-\sigma_{11}^{\mathrm{S}}/v_R^2$.

4.2. General formula for the σ_{33}^{S} and σ_{13}^{S} components

Auld's demonstration [3] may be adapted to $\mathbf{F}$. After reproducing the steps, one shows that:

$$\frac{\Delta v_R}{v_R}(\omega) = -\frac{v_R}{4P}\int_0^\infty \begin{bmatrix} \mathrm{i}\omega\,\mathbf{u} \\ \mathbf{T}_3 \end{bmatrix}^\dagger \Delta\mathbf{F}\begin{bmatrix} \mathbf{T}_3 \\ \mathrm{i}\omega\,\mathbf{u} \end{bmatrix}\mathrm{d}x_3, \tag{14}$$

where † denotes the transpose-conjugate operator and where the displacement $\mathbf{u}$ and normal stress $\mathbf{T}_3$ are approximated by the unperturbed state. Let us focus on the part of $\Delta\mathbf{F}$ that requires this extended formula. To this end, let us separate $\Delta\mathbf{F} = \Delta\mathbf{F}^{\mathrm{S}} + \Delta\mathbf{F}^{33} + \Delta\mathbf{F}^{13}$, $\Delta\mathbf{F}^{\mathrm{S}}$ being the difference between application of Formula (8) with $\rho^{\mathrm{S}}-\sigma_{11}^{\mathrm{S}}/v_R^2$, $\mathbf{A}^{\mathrm{S}}$, $\mathbf{B}^{\mathrm{S}}$, $\mathbf{C}^{\mathrm{S}}$ and the same formula with the unperturbed constants, and $\Delta\mathbf{F}^{33}$, $\Delta\mathbf{F}^{13}$ arising from the modification of the diagonals of $\mathbf{B}$ and $\mathbf{C}$:

$$\Delta\mathbf{F}^{33} = -\frac{\sigma_{33}^{\mathrm{S}}}{v_R^2}\begin{bmatrix} v_R\,\mathbf{B}(\mathbf{C}^{-1})^2 & \mathbf{B}(\mathbf{C}^{-1})^2\mathbf{B}^t \\ v_R^2(\mathbf{C}^{-1})^2 & v_R(\mathbf{C}^{-1})^2\mathbf{B}^t \end{bmatrix}, \tag{15a}$$

$$\Delta\mathbf{F}^{13} = \frac{\sigma_{13}^{\mathrm{S}}}{v_R^2}\begin{bmatrix} v_R\,\mathbf{C}^{-1} & \mathbf{B}\,\mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}^t \\ \mathbf{0} & v_R\,\mathbf{C}^{-1} \end{bmatrix}. \tag{15b}$$

In Eq. (15a) it has been made use of $(\mathbf{C} + \sigma_{33}^{\mathrm{S}}\,\mathbf{I})^{-1} \approx \mathbf{C}^{-1}-\sigma_{33}^{\mathrm{S}}(\mathbf{C}^{-1})^2$, and the $\mathbf{C}$ and $\mathbf{B}$ matrices referred to are built from the unperturbed constants.

In the general case, the dispersion of the Rayleigh wave produced by a static stress can therefore be calculated by applying Eq. (12) with $\Delta\rho = \rho^{\mathrm{S}}-\rho-\sigma_{11}^{\mathrm{S}}/v_R^2$, $\Delta C_{ijkl} = C_{ijkl}^{\mathrm{S}}-C_{ijkl}$, and then adding the extra contribution of σ_{33}^{S} and σ_{13}^{S} by evaluating Eq. (14) with $\Delta\mathbf{F}^{33}$ and $\Delta\mathbf{F}^{13}$.

4.3. Isotropic case

We now apply the theory to a medium which is isotropic in its unperturbed state. The field components of a Rayleigh wave propagating in the x_1 direction are [3]:

$$u_1 = \mathrm{i}\,n_s\left(e^{-n_s\frac{\omega}{v_R}x_3}-\frac{2}{1 + n_s^2}e^{-n_l\frac{\omega}{v_R}x_3}\right)e^{-\mathrm{i}\frac{\omega}{v_R}x_1}, \tag{16a}$$

$$u_2 = 0, \tag{16b}$$

$$u_3 = \left(e^{-n_s\frac{\omega}{v_R}x_3}-\frac{1 + n_s^2}{2}e^{-n_l\frac{\omega}{v_R}x_3}\right)e^{-\mathrm{i}\frac{\omega}{v_R}x_1}, \tag{16c}$$

with $n_s = \sqrt{1-(v_R/v_s)^2}$, $n_l = \sqrt{1-(v_R/v_l)^2}$, and with v_s and v_l being the bulk shear and longitudinal velocities. Note that v_R is such that $4\,n_s\,n_l = (1 + n_s^2)^2$. In a first time, we assume that σ_{33}^{S} and σ_{13}^{S} are null, *i.e.* the perturbed medium may be entirely described by an effective mass density and elastic tensor. Let us rewrite Eq. (12) in the form

$$\frac{\Delta v_R}{v_R}(\omega) = \sum_{\gamma=\rho,\,C_{ij}}\int_0^\infty \Delta\gamma(x_3)f_i^{(\gamma)}\,e^{-n_i\frac{\omega}{v_R}x_3}\frac{\omega}{v_R}\mathrm{d}x_3, \tag{17}$$

where $n_1 = 2n_s$, $n_2 = 2n_l$, $n_3 = n_s + n_l$, and

$$\frac{P}{\omega} = \frac{1}{4}\rho\,v_R^2\,n_s\left(2 + n_s^{-2} + n_l^{-2}-\frac{8}{1 + n_s^2}\right). \tag{18}$$

Because in Eq. (16) the only non-null gradients are ε_{11}, ε_{33} and ε_{13} and because of the $\pi/2$ phase shift between some of these (which leads to terms having zero real parts), it can be seen from Eqs. (12) and (17) that only the $\Delta\gamma = \Delta\rho$, ΔC_{11}, ΔC_{33}, ΔC_{13} and ΔC_{55} contribute to the dispersion. The corresponding $f_i^{(\rho)}$, $f_i^{(C_{11})}$, $f_i^{(C_{33})}$, $f_i^{(C_{13})}$ and $f_i^{(C_{55})}$ constants are

Table 1
Mass density (in $\mathrm{g\,cm^{-3}}$), Lamé's and Murnaghan's constants (in GPa) describing the mild steel [6,8] sample used in Fig. 1, and the Ti-6246 [13] sample used in Figs. 2–4.

	ρ	λ	μ	l	m	n
Steel	7.837	107.4	81.9	−206.5	−600	−800
Ti	4.54	80.0	45.5	−201	−272	−356

given in the Appendix. As can be seen from Eq. (4b) applied to an isotropic medium, C^S_{11}, C^S_{33}, C^S_{13} and C^S_{55} are only affected by the principal static strains ε^S_{11}, ε^S_{22} and ε^S_{33}, from which we deduce that the influence of σ^S_{12} and σ^S_{23} on Δv_R are null at first order. The effect of σ^S_{13} is also zero at first order, because the contribution of $\Delta\mathbf{F}^{13}$ in Eq. (15b) calculated with Eq. (14) only involves products of fields having a $\pi/2$ phase shift, therefore producing a zero real part. We are thus only left with the evaluation of the contribution of $\Delta\mathbf{F}^{33}$ to cover all static stress components. After some tedious but straightforward calculations, Eq. (14) can finally be rewritten in a similar form as Eq. (17):

$$\frac{\Delta v_R}{v_R}(\omega) = \sum_{\gamma=11,22,33} \int_0^\infty \sigma^S_\gamma(x_3) f_i^{(\sigma^S_\gamma)} e^{-n_i \frac{\omega}{v_R} x_3} \frac{\omega}{v_R}\, \mathrm{d}x_3. \tag{19}$$

The $f_i^{(\sigma^S_{11})}$, $f_i^{(\sigma^S_{22})}$ and $f_i^{(\sigma^S_{33})}$ constants appearing in Eq. (19) are given in the Appendix.

Note that because the third order constants do not affect the unperturbed state, some hypothesis can easily be weakened. If $(l, m, n) \to (l, m, n)(x_3)$ are allowed to vary with depth, then $f_i^{(\sigma^S_\gamma)} \to f_i^{(\sigma^S_\gamma)}(x_3)$ are depth-dependent and Eq. (19) is still valid.

As a last comment let us say a word on the high-frequency asymptotic of Eq. (19). Szabo [4] pointed out that the integral operator transforms a series in x_3 into a series in ω^{-1}, and proposed to use this property for an inverse procedure, although he did not deal with static stresses. Ditri and Hongerholt [8] did the same in the context of static stresses, but relied on Husson's formula [7], which is not correct. If we proceed to the polynomial expansion $\sigma^S_\gamma(x_3) = \sum_{j=0} \sigma^S_{\gamma,j} x_3^j$, then:

$$\frac{\Delta v_R}{v_R}(\omega) = \sum_{j=0} \frac{\Phi_j}{\omega^j}, \tag{20a}$$

$$\Phi_j = \frac{1}{v_R} \sum_{\gamma=11,22,33} \sigma^S_{\gamma,j} f_i^{(\sigma^S_\gamma)} j! \left(\frac{v_R}{n_i}\right)^{j+1}. \tag{20b}$$

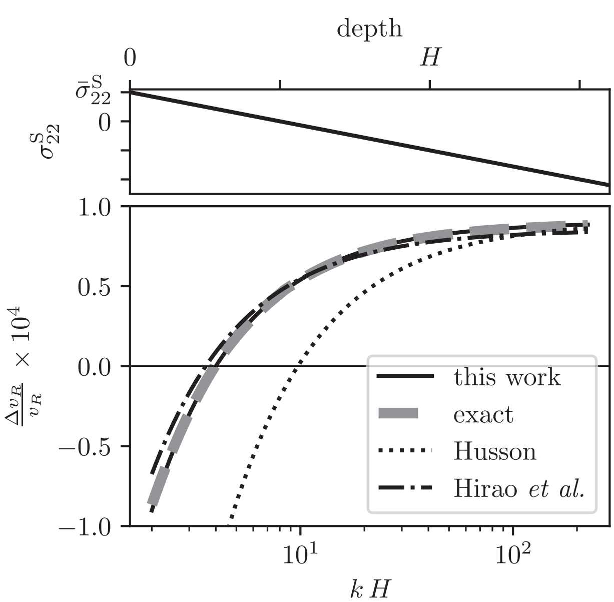

Fig. 1. Dispersion of Rayleigh wave caused by a static stress growing with depth, in a mild steel sample.

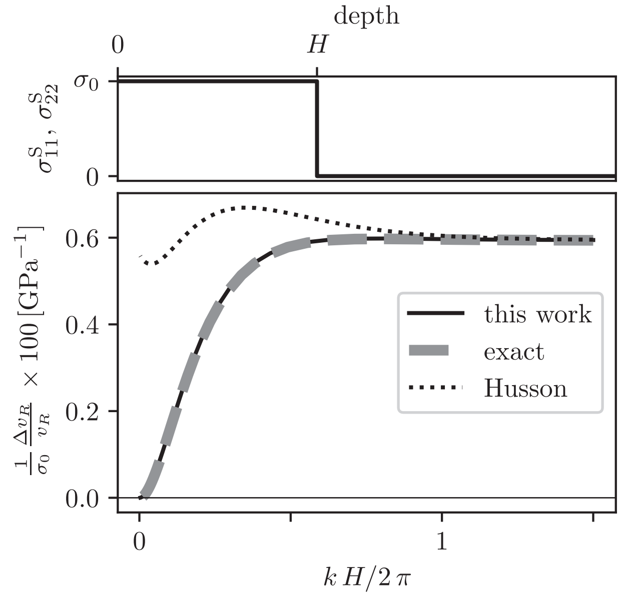

Fig. 2. Dispersion of Rayleigh wave in Ti-6246 produced by a layer of static stress localized at the surface. The stress is isotropic in the (1, 2) plane ($\sigma^S_{11} = \sigma^S_{22}$) and has no vertical component ($\sigma^S_{33} = 0$).

5. Numerical examples

In the following, the perturbation theory derived in this work (see Eq. (19)) is validated against an exact method described earlier [11,12]. Husson's formula [7,8] and, for the first example, the formula of Hirao et al. [6] are also compared. The elastic constants that describe the materials are given in Table 1. All comparisons are made for small values of the static strain ($\varepsilon^S_{ij} \approx 0.1\%$). The range of validity according to this amplitude is not addressed in this work.

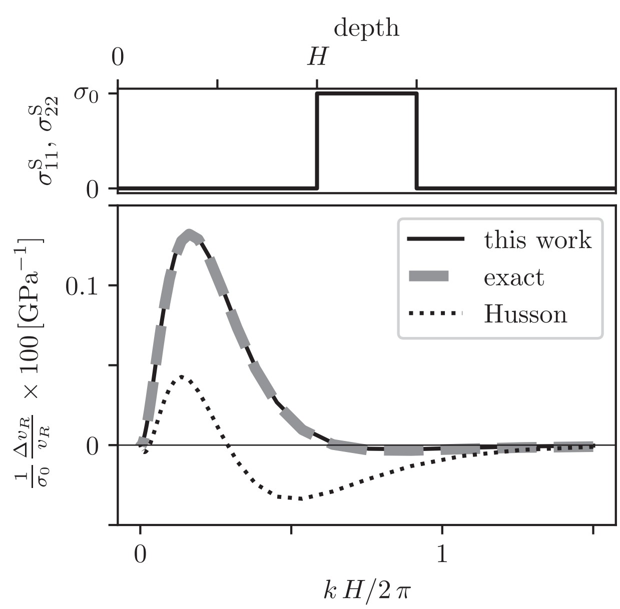

Fig. 3. Dispersion of Rayleigh wave in Ti-6246 produced by a buried layer of static stress. The stress is isotropic in the (1, 2) plane ($\sigma^S_{11} = \sigma^S_{22}$) and has no vertical component ($\sigma^S_{33} = 0$).

5.1. Static stress growing with depth

As a first example, and to compare the aforementioned theories with the set of formulas derived in this work, we re-consider the situation of Hirao et al. These authors applied couples to a mild steel plate in such a way to generate a uni-axial static stress $\sigma_{22}^{S}(x_3) = \overline{\sigma}_{22}^{S}(1-2x_3/H)$, with $\overline{\sigma}_{22}^{S} = 63.75\,[\mathrm{MPa}]$ and $H = 10\,\mathrm{mm}$. They showed that the predicted dispersion was of the form

$$\frac{\Delta v_R}{v_R}(k) = -\left(\beta_0 + \frac{\beta_1}{k\,H}\right)\frac{\nu}{E}\overline{\sigma}_{22}^{S}, \tag{21}$$

and provided the numerical values $\beta_0 = -0.99$ and $\beta_1 = 3.55$. Using Eq. (20b) one can calculate $\beta_0 = -1.05$ and $\beta_1 = 4.19$. The slight difference (0.6%) in β_0 is compatible with a use by Hirao et al. of constants slightly different from those reported, within the numerical precision reported. The discrepancy is however bigger for β_1 and we could not reproduce the value with Eq. (20b) without modifying the elastic constants outside the given numerical precision. We therefore tend to consider it as a numerical hint that the formula that they used is not valid for this example, even if it is difficult to be really categorical (the qualitative demonstration given at the beginning of this article can not be applied to this example because the profile is not localized). In the case of Husson's formula ($\beta_0 = -1.05$ and $\beta_1 = 10.20$), the result is clearly far off for β_1. When using the corrective term indicated in Eq. (1) the results perfectly agree with Eq. (20b).

We present the comparison in Fig. 1. The discrepancies appear at low frequencies. The curve which is labeled "exact" was obtained by finding the root of the determinant of the Transfer Matrix after approximating the stress in thin constant layers. The instabilities that are well known to appear at high frequencies have been assessed by adapting the discretization to the wavelength.

We tried without success to reproduce the results of Hirao et al. by using their formula. One reason could be that their comprehensive set of constants would have been reported with typographical issues. This is why only Husson's formula is compared with the present work in the following examples.

5.2. Static stress localized in a layer

As a second set of examples let us consider stress profiles that are confined to the near-surface region. This choice is made to illustrate situations in which stress is deliberately introduced into the material by a surface treatment, such as shot-peening, laser shock-peening or low plasticity burnishing. Two cases are shown in Figs. 2 and 3, where Husson's formula is compared to Eq. (19) and to an exact calculation. Fig. 2 shows the transition in the variation of velocity from zero to a value proportional to the acoustoelastic coefficient when the frequency is increased. Indeed, the lower frequency limit is only determined by the unchanged substrate, while the wave is confined in a uniform stress region when the penetration is smaller than the depth of the layer. It can be seen that Husson's formula predicts a non-physical lower frequency limit. Fig. 3 shows how band-limited is the variation of velocity caused by a buried layer. Here again, Husson's formula predicts wrong results unless Eq. (1) is used.

5.3. Depth-varying third order constants

As a final set of examples let us consider the dispersion caused by a variation of the third order elastic constants with depth when a static stress is uniformly applied. The aim of this last set of examples is twofold. First, according to a comment we made before, it is the only case where Husson's formula can correctly predict a non-trivial variation of velocity without requiring Eq. (1). The second reason is that this situation seems to have been ignored in the past, while the surface treatments earlier referred to are known to affect the third order elastic constants in the near-surface region as a by-product of plastic deformation.

Fig. 4 shows three sub-figures (a), (b) and (c) corresponding to the three possible directions of the uni-axial and uniformly applied stress. On each sub-figure, three dashed lines are displayed, corresponding to three scenarios of variation of l, m, n by 100% of their value. The curve labeled l refers to a variation $l \to 2l$ while m and n are kept unchanged, and so forth. The variation is confined to a layer of depth H, located at the surface. Because of linearity, any other variation of l, m, n can be obtained by a linear combination of these three scenarios and the reference levels. The dispersion is calculated using Eq. (19), now having $f_i^{(\sigma_\gamma^S)} = f_i^{(\sigma_\gamma^S)}(x_3)$ with the same formal expression (see Appendix). A perfect agreement was found with both the exact method and Husson's formula (in the (a) and (b) cases for which it was designed). It can be seen on Fig. 4 (a), (b) and (c) that none of the situations is sensitive to l. Thus, only $m(x_3)$ and $n(x_3)$ can be expected to be deduced from experimental data, and the inversion would certainly require the effect to be measured in two directions.

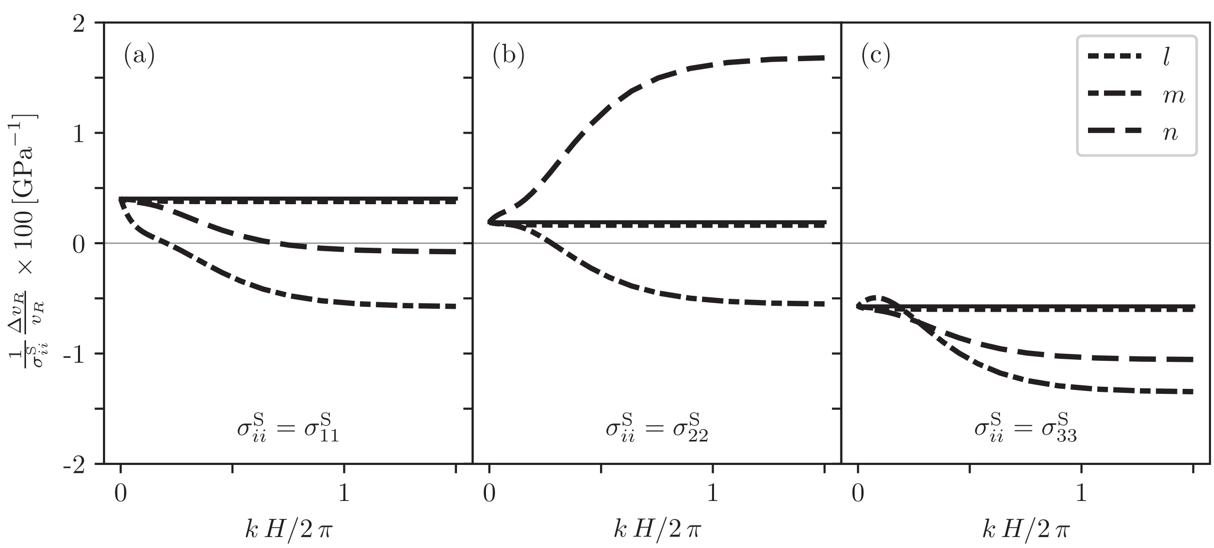

Fig. 4. Dispersion of Rayleigh wave in Ti-6246 produced by depth-varying $(l, m, n) = (l, m, n)_{\mathrm{ref}} + \Delta(l, m, n)(x_3)$, under a uniform uni-axial static stress, (a) σ_{11}^S, (b) σ_{22}^S, (c) σ_{33}^S. The depth-dependency of $\Delta(l, m, n)$ is a layer of thickness H localized at the surface. The continuous curve is the reference, *i.e.* when $\Delta(l, m, n) = 0$. The three dashed curves correspond to three scenarios where one constant is increased by 100% while the other two are kept unchanged.

6. Conclusion

In this work, the acoustoelastic effect on Rayleigh waves under depth-varying loading was addressed via a perturbation approach. Two theories published in the past and widely cited were shown to be different, contrarily to what was claimed. By adapting a demonstration by Auld, a corrected perturbation theory was derived in a general case, covering general anisotropy and all components of loading, and explicit formulas were given for an isotropic medium. Validation against exact computations showed perfect agreement under the hypothesis of low perturbation amplitudes. Numerical examples include depth-varying third order elastic constants, which, to the best of our knowledge, had not been considered before.

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Appendix A. Explicit expressions for isotropic media

ΔC_{ij}^{S} as a function of the static strains ε_i^{S} for an isotropic medium (Eq. (4b)):

$$\Delta C_{11}^{S} = (C_{111} + 3C_{11})\varepsilon_1^{S} + (C_{113} - C_{11})(\varepsilon_2^{S} + \varepsilon_3^{S}),$$
$$\Delta C_{33}^{S} = (C_{111} + 3C_{11})\varepsilon_3^{S} + (C_{113} - C_{11})(\varepsilon_1^{S} + \varepsilon_2^{S}),$$
$$\Delta C_{13}^{S} = (C_{113} + C_{13})(\varepsilon_1^{S} + \varepsilon_3^{S}) + (C_{123} - C_{13})\varepsilon_2^{S},$$
$$\Delta C_{55}^{S} = (C_{155} + C_{55})(\varepsilon_1^{S} + \varepsilon_3^{S}) + (C_{255} - C_{55})\varepsilon_2^{S}.$$

Constants $f_i^{(\rho)} = \frac{v_R^2}{4P/\omega}\overline{f}_i^{(\rho)}$ and $f_i^{(C_{ij})} = \frac{1}{4P/\omega}\widetilde{f}_i^{(C_{ij})}$ appearing in Eq. (17):

$$\overline{f}_1^{(\rho)} = -(1 + n_s^2), \qquad \widetilde{f}_1^{(C_{11})} = n_s^2,$$
$$\overline{f}_2^{(\rho)} = -\frac{n_s}{n_l}(1 + n_l^2), \qquad \widetilde{f}_2^{(C_{11})} = \frac{n_s}{n_l},$$
$$\overline{f}_3^{(\rho)} = 4n_s\frac{n_s + n_l}{1 + n_s^2}, \qquad \widetilde{f}_3^{(C_{11})} = -\frac{n_s}{n_l}(1 + n_s^2),$$
$$\widetilde{f}_1^{(C_{33})} = n_s^2, \qquad \widetilde{f}_1^{(C_{13})} = -2n_s^2,$$
$$\widetilde{f}_2^{(C_{33})} = n_s n_l^3, \qquad \widetilde{f}_2^{(C_{13})} = -\frac{1}{2}(1 + n_s^2)^2,$$
$$\widetilde{f}_3^{(C_{33})} = -\frac{1}{4}(1 + n_s^2)^3, \qquad \widetilde{f}_3^{(C_{13})} = 4n_s^2\frac{1 + n_l^2}{1 + n_s^2},$$
$$\widetilde{f}_1^{(C_{55})} = (1 + n_s^2)^2,$$
$$\widetilde{f}_2^{(C_{55})} = (1 + n_s^2)^2,$$
$$\widetilde{f}_3^{(C_{55})} = -2(1 + n_s^2)^2.$$

Constants $f_i^{(\sigma_{jj}^{S})} = \frac{1}{4P/\omega}\widetilde{g}_i^{(\sigma_{jj}^{S})} + \sum_{\gamma=\rho, C_{ij}} \alpha_{(\gamma)}^{(\sigma_{jj}^{S})} f_i^{(\gamma)}$ appearing in Eq. (19):

$$\alpha_{(\rho)}^{(\sigma_{11}^{S})} = \alpha_{(\rho)}^{(\sigma_{22}^{S})} = \alpha_{(\rho)}^{(\sigma_{33}^{S})} = -\frac{\rho(1 - 2\nu)}{E},$$
$$\alpha_{(C_{11})}^{(\sigma_{11}^{S})} = \alpha_{(C_{33})}^{(\sigma_{33}^{S})} = \frac{1}{E}[(C_{111} + 3C_{11}) - 2\nu(C_{113} - C_{11})],$$
$$\alpha_{(C_{33})}^{(\sigma_{11}^{S})} = \alpha_{(C_{11})}^{(\sigma_{33}^{S})} = \alpha_{(C_{11})}^{(\sigma_{22}^{S})} = \alpha_{(C_{33})}^{(\sigma_{22}^{S})} = \frac{1}{E}[(1 - \nu)(C_{113} - C_{11}) - \nu(C_{111} + 3C_{11})],$$
$$\alpha_{(C_{13})}^{(\sigma_{11}^{S})} = \alpha_{(C_{13})}^{(\sigma_{33}^{S})} = \frac{1}{E}[(1 - \nu)(C_{113} + C_{13}) - \nu(C_{123} - C_{13})],$$
$$\alpha_{(C_{55})}^{(\sigma_{11}^{S})} = \alpha_{(C_{55})}^{(\sigma_{33}^{S})} = \frac{1}{E}[(1 - \nu)(C_{155} + C_{55}) - \nu(C_{255} - C_{55})],$$
$$\alpha_{(C_{13})}^{(\sigma_{22}^{S})} = \frac{1}{E}[(C_{123} - C_{13}) - 2\nu(C_{113} + C_{13})],$$
$$\alpha_{(C_{55})}^{(\sigma_{22}^{S})} = \frac{1}{E}[(C_{255} - C_{55}) - 2\nu(C_{155} + C_{55})],$$

and:

$$\widetilde{g}_i^{(\sigma_{11}^S)} = -v_R^2 \overline{f}_i^{(\rho)},$$

$$\widetilde{g}_i^{(\sigma_{22}^S)} = 0,$$

$$\begin{aligned}\widetilde{g}_1^{(\sigma_{33}^S)} &= \frac{C_{13}}{C_{11}}\widetilde{f}_1^{(C_{13})} - \left(\frac{C_{13}}{C_{11}}\right)^2\widetilde{f}_1^{(C_{11})} + \widetilde{f}_1^{(C_{33})} + \widetilde{f}_1^{(C_{55})}\\ &\quad + n_s^2\left[\left(\frac{C_{13}}{C_{11}}+1\right)^2-3\right]-1,\end{aligned}$$

$$\begin{aligned}\widetilde{g}_2^{(\sigma_{33}^S)} &= \frac{C_{13}}{C_{11}}\widetilde{f}_2^{(C_{13})} - \left(\frac{C_{13}}{C_{11}}\right)^2\widetilde{f}_2^{(C_{11})} + \widetilde{f}_2^{(C_{33})} + \widetilde{f}_2^{(C_{55})}\\ &\quad + n_s n_l\left[\frac{1}{n_l^2}\left(\frac{C_{13}}{C_{11}}\right)^2 + 2\frac{C_{13}}{C_{11}}-3\right],\end{aligned}$$

$$\begin{aligned}\widetilde{g}_3^{(\sigma_{33}^S)} &= \frac{C_{13}}{C_{11}}\widetilde{f}_3^{(C_{13})} - \left(\frac{C_{13}}{C_{11}}\right)^2\widetilde{f}_3^{(C_{11})} + \widetilde{f}_3^{(C_{33})} + \widetilde{f}_3^{(C_{55})}\\ &\quad - (1+n_s^2)\left[\frac{n_s}{n_l}\left(\frac{C_{13}}{C_{11}}\right)^2 + n_s\frac{C_{13}}{C_{11}}(n_l+\frac{1}{n_l})\right.\\ &\quad \left. - n_s^2-2\right].\end{aligned}$$

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