

On the Rayleigh-geometric distribution with applications

Idika E. Okorie ^{a,*}, Anthony C. Akpanta ^b, Johnson Ohakwe ^c, David C. Chikezie ^b,
Chris U. Onyemachi ^b

^a School of Mathematics, University of Manchester, Manchester, M13 9PL, UK

^b Department of Statistics, Abia State University, Uturu, Abia State, Nigeria

^c Department of Mathematics & Statistics, Faculty of Sciences, Federal University Otuoke, Bayelsa State, P.M.B 126 Yenagoa, Bayelsa, Nigeria

ARTICLE INFO

Keywords:

Mathematics
Hazard rate
Rayleigh distribution
Geometric distribution
Queueing analysis
Maximum likelihood

ABSTRACT

A two-parameter Rayleigh-geometric distribution with increasing-decreasing-increasing and strictly increasing hazard rate characteristics is reviewed. Various properties are discussed and expressed analytically. The estimation of the distribution parameters is studied by the method of maximum likelihood and validated by a simulation study. Numerical examples based on two real data-sets on the waiting time in queue and CO₂ emissions are given. The Rayleigh-geometric distribution in this paper has a simpler analytical expression compared to the pre-existing distributions with different parameterizations.

1. Introduction

In recent times, there is increasing attempt by several researchers from different academic spheres to define new probability distributions for appropriate modeling of various complex real-life phenomena. For instance, Adamidis and Loukas (1998), Kuş (2007), Ristić and Nadarajah (2014), Tahir et al. (2016), Okorie et al. (2017a), Okorie et al. (2017b), and so on.

Some of the new probability distributions are quite flexible in that they result in some other well-defined probability distributions when their parameter(s) are set to certain values. For example, Mahmoudi and Shiran (2012) introduced a four-parameter distribution known as the exponentiated Weibull-geometric (EWG) distribution. The EWG distribution is a compound mixture of the exponentiated Weibull (EW) distribution due to Mudholkar and Srivastava (1993) and the geometric distribution. The EWG distribution nest a couple of other distributions as special cases and they include the generalized exponential-geometric (GEG), complementary Weibull-geometric (CWG), complementary exponential-geometric (CEG), exponentiated Rayleigh-geometric (ERG) and, the Rayleigh-geometric (RG) distributions. The probability density function (PDF) of the EWG distribution is given by

$$f_X(x) = \frac{(1-\theta)\alpha\gamma\beta^\gamma x^{\gamma-1} \exp(-[\beta x]^\gamma)(1-\exp[-(\beta x)^\gamma])^{\alpha-1}}{[1-\theta(1-\exp[-(\beta x)^\gamma])^\alpha]^2}$$

and the cumulative distribution function (CDF) is given by

$$F_X(x) = \frac{(1-\theta)(1-\exp[-(\beta x)^\gamma])^\alpha}{1-\theta(1-\exp[-(\beta x)^\gamma])^\alpha},$$

where $\alpha, \beta, \gamma > 0$, and $\theta \in [0, 1)$.

Particularly, the EWG distribution reduces to the RG distribution when $\alpha = 1$ and $\gamma = 2$. The PDF of the RG distribution is given by

$$f_X(x) = \frac{2(1-\theta)\beta^2 x \exp[-(\beta x)^2]}{[1-\theta(1-\exp[-(\beta x)^2])]^2},$$

with CDF

$$F_X(x) = \frac{(1-\theta)(1-\exp[-(\beta x)^2])}{1-\theta(1-\exp[-(\beta x)^2])},$$

where $\beta > 0$, and $\theta \in [0, 1)$.

This paper is devoted to studying a probability distribution that had appeared in previous studies in slightly different forms. The distribution in its present form is simpler in expression and it has among other appealing characteristics an increasing hazard rate. The increasing hazard rate is a popular and useful concept in life testing and so many other

* Corresponding author.

E-mail address: idika.okorie@manchester.ac.uk (I.E. Okorie).

areas of applied probability and statistics, for example; many real-life phenomena particularly those that are related to electronic components, devices, and machines exhibit the increasing hazard rate property due to regular use and wear. Moreover, the distribution under study is identified to be suitable for modeling queueing and CO₂ emissions data.

The remaining part of this paper is in the following order. Section 2 introduces the distribution, defines some related functions like the reliability and hazard rate functions, and discusses some limit behaviors, Section 3 gives some basic properties of the distribution, Section 4 presents the estimation of parameters, Section 5 illustrates the usefulness and flexibility of the distribution by two real data-sets, and Section 6 gives the concluding remarks.

2. Model

Suppose there are Q functional components of a working system operating independently, each with failure time denoted by Y_1, Y_2, \dots, Y_Q , and the components are arranged in series; the system stops working when one of the components collapses. Assuming that the failure time of the components Y follows the Rayleigh distribution with PDF

$$f_Y(y) = y/\sigma^2 \exp(-y^2/[2\sigma^2]), y \in \mathbb{R}_+, \sigma > 0$$

and CDF

$$F_Y(y) = 1 - \exp(-y^2/[2\sigma^2]), y \in \mathbb{R}_+, \sigma > 0,$$

and the discrete random variable Q follows the geometric distribution with probability mass function $\mathbb{P}_Q(q) = (1 - p)p^{q-1}$, for $q \in \mathbb{N}$ and $p \in (0, 1)$, then the failure time X of the system is given by $X = \min(Y_1, Y_2, \dots, Y_Q) = \min \prod_{i=1}^Q Y_i$. Hence, the conditional CDF of $X|Q = q$ is given by

$$\begin{aligned} F_{X|Q=q}(x|q) &= 1 - [\mathbb{P}(Y_1 > x)]^q \\ &= 1 - [1 - \mathbb{P}(Y_1 \leq x)]^q \\ &= 1 - \exp(-x^2 q/[2\sigma^2]) \end{aligned}$$

and the marginal CDF of X is given by

$$F_X(x) = \frac{1 - \exp\left\{-\frac{x^2}{2\sigma^2}\right\}}{1 - p \exp\left\{-\frac{x^2}{2\sigma^2}\right\}}; x \in \mathbb{R}_+, \sigma > 0, p \in (0, 1). \tag{1}$$

The PDF of X is given by

$$f_X(x) = \frac{(1 - p)x \exp\left\{-\frac{x^2}{2\sigma^2}\right\}}{\sigma^2 \left(1 - p \exp\left\{-\frac{x^2}{2\sigma^2}\right\}\right)^2}; x \in \mathbb{R}_+, \sigma > 0, p \in (0, 1). \tag{2}$$

The corresponding reliability function and hazard rate function of X are given by

$$R_X(x) = \frac{(1 - p) \exp\left\{-\frac{x^2}{2\sigma^2}\right\}}{1 - p \exp\left\{-\frac{x^2}{2\sigma^2}\right\}}; x \in \mathbb{R}_+, \sigma > 0, p \in (0, 1)$$

and

$$h_X(x) = \frac{x}{\sigma^2 \left(1 - p \exp\left\{-\frac{x^2}{2\sigma^2}\right\}\right)}; x \in \mathbb{R}_+, \sigma > 0, p \in (0, 1)$$

respectively.

The two-parameter distribution whose CDF and PDF are given by Equations (1) and (2) is called the Rayleigh-geometric distribution (RGD for brevity) where σ is the scale parameter and p is the shape parameter. The RGD is a special case of the geometric generalized family of distributions and the physical interpretation of the exponential-geometric distribution (EGD) due to Adamidis and Loukas (1998) can be extended to the RGD. The PDF of the RGD is always unimodal and

any p with mode $\neq 0$ and the hazard rate function is either increasing-decreasing-increasing (if $p \rightarrow 1$) or strictly increasing (if $p \rightarrow 0$). The Rayleigh distribution with parameter σ is the limiting case of the RGD when $p \rightarrow 0$. The limiting behavior of the PDF and the hazard rate function are $f_X(0) = f_X(\infty) = 0$ and $h_X(0) = 0$ and $h_X(\infty) \sim x/\sigma^2$. The plots of the PDF and hazard rate function of the RGD for fixed σ and different values of p are shown in Fig. 1.

2.1. Relationship with other distributions

To the best of our knowledge, there is no paper focusing on either the direct derivation of the RGD or dedicated to the discussion of its properties and inferential issues as in this paper but, as we mentioned earlier, the RGD has apparently emerged in the literature as a special case of some well-known distributions. For instance:

1. The Marshall-Olkin Weibull distribution due to Marshall and Olkin (1997) becomes the RGD, when $\alpha = 1 - p$ and $\lambda^\beta = \frac{1}{2\sigma^2}$.
2. The Marshall-Olkin Rayleigh distribution due to Ozel and Cakmakyapan (2015) is equivalent to the RGD, when $\gamma = 1 - p$.
3. The Weibull-geometric distribution reduces to the RGD for $\alpha = 2$ and $\beta = \frac{1}{2\sigma}$ (see Hamedani and Ahsanullah, 2011).
4. The Weibull-geometric distribution (WGD) (Barreto-Souza et al., 2011) becomes the RGD, when $\alpha = 2$ and $\beta = \frac{1}{\sigma} \sqrt{\frac{1}{2}}$.
5. The generalized linear failure rate-geometric (GLFRG) distribution due to Nadarajah et al. (2014) boils down to the RGD for $a = 0$, $\alpha = 1$, and $b = \frac{1}{\sigma^2}$.

Notably, the RGD is related to the EGD due to Adamidis and Loukas (1998) in the following way; suppose X follows the RGD and $y = \frac{x^2}{2\sigma^2}$, then Y follows the EGD with parameters $p \in (0, 1)$ and $\beta = 1$.

The RGD is different from and has no link with the Rayleigh-geometric distribution due to Mahmoudi and Shiran (2012).

3. Theory

3.1. Quantile function and random number generation

Apart from generating lookup tables for fractiles, the quantile function has also been used to study several mathematical properties of a probability distribution.

Theorem 3.1. *If $X \sim \text{RGD}$, then the T th quantile function of X is given by*

$$X(T) = \sigma \sqrt{-2 \log \left(\frac{1 - T}{1 - Tp} \right)}; \sigma > 0, p, T \in (0, 1).$$

Corollary 3.1.1. *If $X \sim \text{RGD}$, then the median M of X is given by*

$$\text{median} = \sigma \sqrt{2 \log(2 - p)}; \sigma > 0, p \in (0, 1).$$

Corollary 3.1.2. *If $T \sim U(0, 1)$, then $X(T) \sim \text{RGD}$.*

Corollary 3.1.3. *An alternative to the classical measure of skewness and kurtosis due to Galton (1911) and Moors (1988), respectively is based on the quantile function and they are known as the Galton's skewness and Moors' kurtosis.*

The Galton's skewness and Moors' kurtosis coefficient are given by

$$S_{kw} = \frac{X(6/8) - 2X(4/8) + X(2/8)}{X(6/8) - X(2/8)}$$

and

$$k_{ut} = \frac{X(7/8) - X(5/8) + X(3/8) - X(1/8)}{X(6/8) - X(2/8)}$$

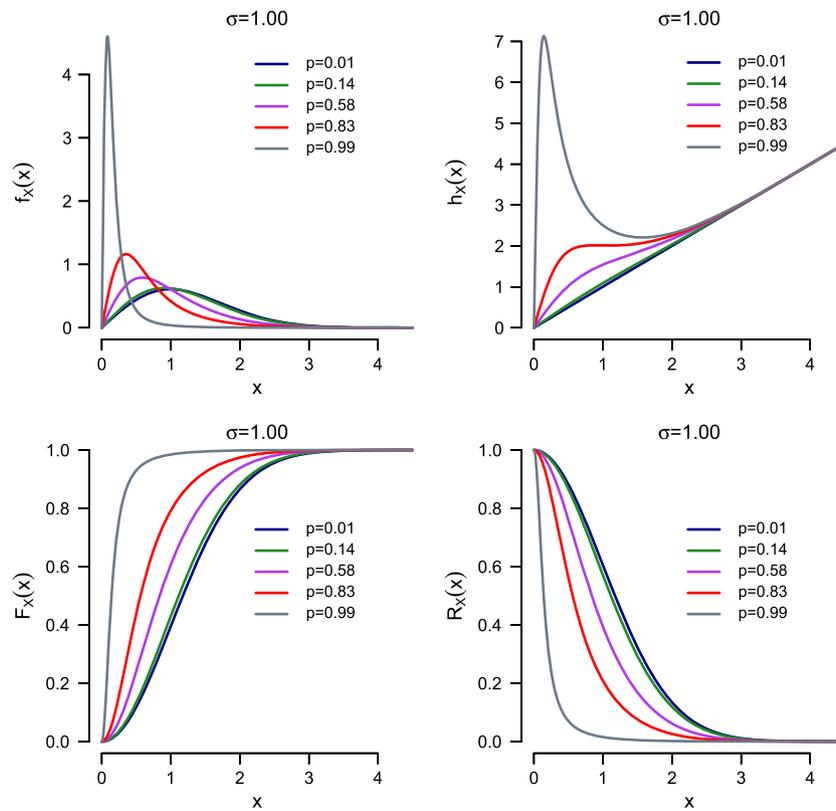


Fig. 1. Plots of the PDF (top-left), hazard rate function (top-right), CDF (bottom-left), and reliability function (bottom-right) for selected parameter values of the RGD.

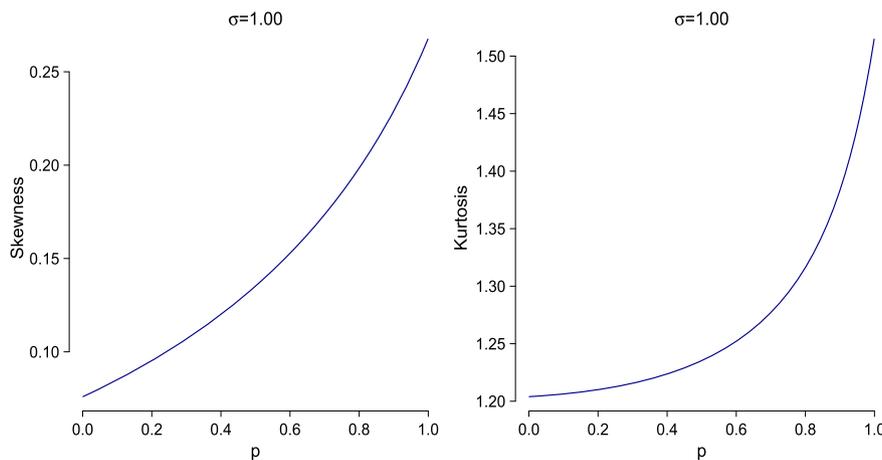


Fig. 2. Plots of the Galton's skewness (left) and Moors' kurtosis (right) of the RGD for fixed scale parameter σ and different values of $p \in (0, 1)$.

respectively. Fig. 2 show the plots of S_{kw} (left) and k_{ut} (right) for fixed σ and different values of p of the RGD. It is clear from Fig. 2 that the RGD is always right skewed and leptokurtic.

3.2. Useful expansions

Mixture representations of Equations (1) and (2) are required to obtain the properties of the RGD, the representations are based on the generalized binomial series and this idea is used throughout the remaining sections. For $|y| < 1$ and $\zeta > 0$, we have

$$(1 - y)^{-\zeta} = \sum_{\ell=0}^{\infty} \frac{\Gamma(\zeta + \ell)}{\Gamma(\ell + 1)\Gamma(\zeta)} y^\ell, \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function. Rewriting Equations (1) and (2) in terms of Equation (3) we obtain

$$F_X(x) = \sum_{i=0}^{\infty} p^i \exp\left\{-\frac{x^2}{2\sigma^2}i\right\} - \sum_{i=0}^{\infty} p^i \exp\left\{-\frac{x^2}{2\sigma^2}[1+i]\right\} \tag{4}$$

and

$$f_X(x) = \frac{1-p}{\sigma^2} \sum_{i=0}^{\infty} \frac{\Gamma(2+i)}{\Gamma(1+i)} p^i x \exp\left\{-\frac{x^2}{2\sigma^2}[1+i]\right\} \tag{5}$$

respectively. Equations (4) and (5) give the CDF and PDF of the RGD as a mixture of Raleigh and geometric distribution.

Theorem 3.2. If $X \sim \text{RGD}$, then the k th ordinary moment of X is given by Equation (6).

Table 1
Some numerical values of the mean and variance of the RGD.

	[0.12,4.50]	[0.38,1.00]	[0.63,8.00]	[p,σ] ↓ [0.94,0.30]	[0.08,1.70]	[0.01,0.01]
Mean	5.430327	1.083086	7.304338	0.136269	2.078852	0.012496
Variance	8.478062	0.386830	21.389170	0.013755	1.220752	0.000043

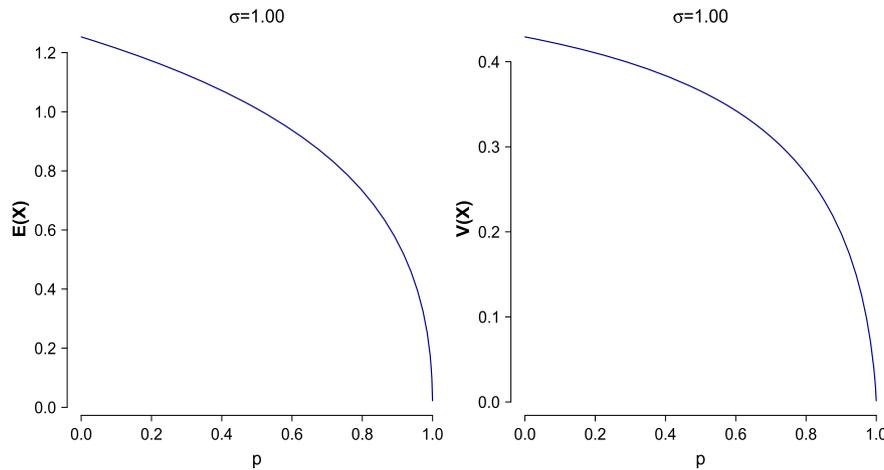


Fig. 3. Plots of the mean (left) and variance (right) of the RGD for fixed scale parameter σ and different values of $p \in (0, 1)$.

$$\begin{aligned}
 E(X^k) &= \int_{\text{all } x} x^k f_X(x) dx; k \in \mathbb{N} \\
 &= \frac{1-p}{\sigma^2} \sum_{i=0}^{\infty} \frac{\Gamma(2+i)}{\Gamma(1+i)} p^i \int_0^{\infty} x^{k+1} \exp\left\{-\frac{x^2}{2\sigma^2}[1+i]\right\} dx \\
 &= [1-p](2\sigma^2)^{\frac{k}{2}} \Gamma\left(\frac{k}{2}+1\right) \sum_{i=0}^{\infty} \frac{\Gamma(2+i)}{\Gamma(1+i)} \frac{p^i}{[1+i]^{\frac{k}{2}+1}}. \tag{6}
 \end{aligned}$$

The third identity of Equation (6) was obtained by substituting for $\frac{x^2}{2\sigma^2}[1+i]$ and simplifying. If k is an even number, Equation (6) can be re-written in terms of the Wright's generalized hypergeometric function as

$$\begin{aligned}
 E(X^k) &= [1-p](2\sigma^2)^{\frac{k}{2}} \Gamma\left(\frac{k}{2}+1\right) \sum_{i=0}^{\infty} \frac{p^i}{i!} \frac{\Gamma(2+i)}{(1+i)^{\frac{k}{2}+1}} \\
 &= [1-p](2\sigma^2)^{\frac{k}{2}} \Gamma\left(\frac{k}{2}+1\right) \sum_{i=0}^{\infty} \frac{p^i}{i!} \frac{\Gamma(2+i)\Gamma(1+i)\cdots\Gamma(1+i)}{\Gamma(2+i)\cdots\Gamma(2+i)} \\
 &= [1-p](2\sigma^2)^{\frac{k}{2}} \Gamma\left(\frac{k}{2}+1\right)^{\frac{k}{2}+2} \Psi_{\frac{k}{2}+1} \left[\begin{matrix} (2, 1), (1, 1), \dots, (1, 1) \\ (2, 1), \dots, (2, 1) \end{matrix} \middle| p \right], \tag{7}
 \end{aligned}$$

where ${}_p\Psi_q(\cdot)$ in Equation (7) is the Wright's generalized hypergeometric function with $\Gamma(1+i)$ and $\Gamma(2+i)$ multiplied $\frac{k}{2}+1$ times. The Wright's generalized hypergeometric function is defined by

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, C_1), \dots, (\alpha_p, C_p) \\ (\beta_1, D_1), \dots, (\beta_q, D_q) \end{matrix} \middle| z \right] = \sum_{i=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(\alpha_j + iC_j)}{\prod_{j=1}^q \Gamma(\beta_j + iD_j)} \frac{z^i}{i!}$$

for $-\infty < z < \infty$, where $-\infty < \alpha_j < \infty$, $-\infty < \beta_j < \infty$, $C_j \neq 0$, $D_j \neq 0$ for $j = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$; see Wright (1935), Mathai and Saxena (1978), Srivastava et al. (1982) and Kilbas et al. (2006, Equation (1.9)).

Corollary 3.2.1. The mean $E(X)$ of the RGD could be obtained from Equation (6) by setting k to 1 and other higher order moments can equally be obtained by the appropriate substitution of k . The variance $V(X)$ of the RGD could be obtained by evaluating $E(X^2) - [E(X)]^2$.

Numerical values of the mean and variance of the RGD are presented in Table 1 and Fig. 3 for various parameter combinations. Fig. 3 indicates that the mean and variance of the RGD decrease with respect to the increasing values of p and the mean is always greater than the variance for different parameter combinations of σ and p .

Corollary 3.2.2. If $X \sim \text{RGD}$ then the moment generating function of X can be obtained by substituting $E(X^k)$ in Equations (6) or (7) into Equation (8).

$$\begin{aligned}
 M_X(t) &= E[\exp(tx)] = \int_{\text{all } x} \exp(tx) f_X(x) dx = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^{\infty} x^k f_X(x) dx \\
 &= \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k). \tag{8}
 \end{aligned}$$

3.3. Mean residual life

The mean residual life of a continuous random variable X say, denoted by $\mu_{rl}(x)$ is a very useful tool in life testing situations for describing the ageing process of X . The $\mu_{rl}(x)$ gives the expected remaining lifetime of a component or device that has survived up to time x .

Theorem 3.3. If $X \sim \text{RGD}$, then the mean residual life of X is given by Equation (9).

$$\begin{aligned}
 \mu_{rl}(x) &= \frac{1}{R_X(x)} \int_x^{\infty} R_T(t) dt, x > 0 \\
 &= \frac{1}{R_X(x)} \int_x^{\infty} \left([1-p] \exp\left\{-\frac{t^2}{2\sigma^2}\right\} \right) \left(1 - p \exp\left\{-\frac{t^2}{2\sigma^2}\right\} \right)^{-1} dt \\
 &= \frac{1}{R_X(x)} \left\{ \sum_{i=0}^{\infty} p^i \int_x^{\infty} \exp\left\{-\frac{t^2}{2\sigma^2}[1+i]\right\} dt \right. \\
 &\quad \left. - \sum_{i=0}^{\infty} p^{1+i} \int_x^{\infty} \exp\left\{-\frac{t^2}{2\sigma^2}[1+i]\right\} dt \right\}
 \end{aligned}$$

$$= \frac{\sigma^2}{x} \left(1 - p \exp \left\{ -\frac{x^2}{2\sigma^2} \right\} \right) \sum_{i=0}^{\infty} \frac{p^i \exp \left\{ -\frac{x^2}{2\sigma^2} i \right\}}{1+i}, \tag{9}$$

where $R_X(x)$ denote the reliability function of the RGD in Section 2. The third identity of Equation (9) is the expanded form of the second identity expressed in terms of Equation (3) and the fourth identity was obtained after substituting for $\frac{x^2}{2\sigma^2} [1+i]$ in the third identity and simplifying.

3.4. Mean waiting time

The mean waiting time of a continuous random variable X say, denoted by $\mu_{wt}(x)$ is a very useful tool in reliability, survival analysis and actuarial science for describing several lifetime phenomena. Suppose we just noticed at time x that a component or device had failed, $\mu_{wt}(x)$ can be used to calculate the exact failure time of X .

Theorem 3.4. *If $X \sim \text{RGD}$, then the mean waiting time of X is given by Equation (10).*

$$\begin{aligned} \mu_{wt}(x) &= \frac{1}{F_X(x)} \int_0^x F_T(t) dt; x > 0 \\ &= \frac{1}{F_X(x)} \sum_{i=0}^{\infty} p^i \int_0^x \exp \left\{ -\frac{t^2}{2\sigma^2} i \right\} dt \\ &\quad - \frac{1}{F_X(x)} \sum_{i=0}^{\infty} p^i \int_0^x \exp \left\{ -\frac{t^2}{2\sigma^2} [1+i] \right\} dt \\ &= \frac{\sigma\sqrt{2}}{2F_X(x)} \sum_{i=0}^{\infty} p^i \left\{ \frac{\gamma \left(\frac{1}{2}, \frac{x^2}{2\sigma^2} i \right)}{\sqrt{i}} - \frac{\gamma \left(\frac{1}{2}, \frac{x^2}{2\sigma^2} [1+i] \right)}{\sqrt{1+i}} \right\}, \tag{10} \end{aligned}$$

where $F_X(x)$ is the CDF of the RGD in Equation (1) and $\gamma(\cdot, \cdot)$ is the lower incomplete gamma function. The third identity of Equation (10) was obtained after substituting $\frac{t^2}{2\sigma^2} i$ and $\frac{t^2}{2\sigma^2} [1+i]$ in the second identity and performing some algebra.

3.5. Mode

Like the mean and median, the mode is used to describe the distribution of a random variable. The mode of a continuous random variable X say, is the value of X at which its PDF has a local maximum value. The local maximum value of the RGD is obtained as the solution of

$$\frac{d \log f_X(x)}{dx} = \frac{1}{x} + \frac{1}{\sigma^2} \left\{ -1 - \frac{2p}{\exp \left\{ -\frac{x^2}{2\sigma^2} \right\} - p} \right\} x = 0. \tag{11}$$

Equation (11) does not have any analytical solution in terms of x and must be calculated numerically. Some of the numerical values of the mode of the RGD are calculated for a fixed value of σ and varying values of p by the `uniroot` function in R and reported in Table 2.

The result in Table 2 shows that the mode of the RGD is always within the interval of $(0, 1)$ and decreases with the increasing values of p .

3.6. Mean deviations

The mean and median are well-known measures of central tendency hence, the degree of spread in a population can reasonably be quantified either by the mean of the distances between the observations and the mean μ or median M . The mean deviations about the mean Ω_1 and the mean deviations about the median Ω_2 can be defined as

Table 2

Some numerical values of the mode of the RGD.

	[0.01,1.00]	[0.14,1.00]	[0.58,1.00]	[0.83,1.00]	[0.99,1.00]
Mode	0.9879725	0.8449239	0.4905438	0.3018392	0.0779182

$$\Omega_1(x) = \int_0^{\infty} |x - \mu| f_X(x) dx = 2[\mu F_X(\mu) - \Psi(\mu)]$$

and

$$\Omega_2(x) = \int_0^{\infty} |x - M| f_X(x) dx = \mu - 2\Psi(M)$$

respectively.

If the random variable X say, follows the RGD and $\exists d \in (0, \infty)$ then $\Psi(\cdot)$ is given by

$$\Psi(d) = \int_0^d x f_X(x) dx = \sigma\sqrt{2}(1-p) \sum_{i=0}^{\infty} \frac{p^i}{[1+i]^{\frac{1}{2}}} \gamma \left(\frac{3}{2}, \frac{d^2[1+i]}{2\sigma^2} \right), \tag{12}$$

where M is given in Corollary 3.1.1 and μ is obtained by either setting $k = 1$ in Equation (6) or Equation (7).

3.7. Lorenz and Bonferroni curves

The application of Lorenz and Bonferroni curves are popularly known in Economics as indices for measuring inequality in terms of income and poverty distribution in a given population and they have also gained widespread utility in reliability engineering, actuarial science, and medicine. The Lorenz and Bonferroni curves for the RGD are given by

$$L(t) = \frac{1}{\mu} \int_0^q x f_X(x) dx = \frac{\Psi(q)}{\mu} = \frac{1}{\Gamma \left(\frac{3}{2} \right)} \sum_{i=0}^{\infty} p^i \gamma \left(\frac{3}{2}, \frac{q^2[1+i]}{2\sigma^2} \right) \tag{13}$$

and

$$B(t) = \frac{L(t)}{t} = \frac{1}{t\Gamma \left(\frac{3}{2} \right)} \sum_{i=0}^{\infty} p^i \gamma \left(\frac{3}{2}, \frac{q^2[1+i]}{2\sigma^2} \right) \tag{14}$$

respectively, where $q = G^{-1}(t)$, $t \in (0, 1)$ and μ is obtained by either setting $k = 1$ in Equation (6) or Equation (7).

We omit the detail for the derivations of Equation (12), (13), and (14) because they are similar to that of Equation (6).

4. Calculation

Let $X \sim \text{RGD}$ and $\Theta = (p, \sigma)'$ be the parameter vector. The log-likelihood function $\mathcal{L} = \mathcal{L}(\Theta)$ based on a random sample of size n is given by

$$\begin{aligned} \mathcal{L}(\Theta) &= -2n \log(\sigma) + n \log(1-p) + \sum_{i=1}^n \log(x_i) \\ &\quad - \frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2 - 2 \sum_{i=1}^n \log \left(1 - p \exp \left\{ -\frac{x_i^2}{2\sigma^2} \right\} \right) \end{aligned}$$

The partial derivative of $\mathcal{L}(\Theta)$ with respect to the parameters are

$$\frac{\partial \mathcal{L}(\Theta)}{\partial p} = -\frac{n}{1-p} + 2 \sum_{i=1}^n \frac{\exp \left\{ -\frac{x_i^2}{2\sigma^2} \right\}}{1 - p \exp \left\{ -\frac{x_i^2}{2\sigma^2} \right\}},$$

and

Table 3
Simulation results.

Actual values		MLEs									
p	σ	n	$\hat{\beta}$	$\hat{\sigma}$	$SE_{\hat{\beta}}$	$SE_{\hat{\sigma}}$	$Bias_{\hat{\beta}}$	$Bias_{\hat{\sigma}}$	$MSE_{\hat{\beta}}$	$MSE_{\hat{\sigma}}$	
0.12	4.5	10	0.1890	4.6544	0.2811	1.1245	0.0690	0.1544	0.0837	1.2870	
		25	0.1773	4.6121	0.2384	0.7244	0.0573	0.1121	0.0600	0.5368	
		50	0.1677	4.6142	0.2115	0.5205	0.0477	0.1142	0.0469	0.2837	
		75	0.1552	4.5805	0.1857	0.4386	0.0352	0.0805	0.0357	0.1986	
		100	0.1484	4.5699	0.1727	0.3896	0.0284	0.0699	0.0306	0.1566	
		150	0.1451	4.5626	0.1573	0.3251	0.0251	0.0626	0.0254	0.1095	
		200	0.1251	4.5159	0.1354	0.2533	0.0051	0.0159	0.0183	0.0644	
		250	0.1354	4.5429	0.1308	0.2427	0.0154	0.0429	0.0173	0.0607	
		300	0.1261	4.5265	0.1235	0.2237	0.0061	0.0265	0.0153	0.0507	
		0.38	1.00	10	0.2604	1.0142	0.3191	0.3073	-0.1196	0.0142	0.1160
25	0.2895			1.0001	0.2814	0.1934	-0.0905	0.0001	0.0873	0.0374	
50	0.3137			1.0025	0.2520	0.1637	-0.0663	0.0025	0.0679	0.0268	
75	0.3241			0.9975	0.2278	0.1315	-0.0559	-0.0025	0.0550	0.0173	
100	0.3323			0.9965	0.2020	0.1100	-0.0477	-0.0035	0.0430	0.0121	
150	0.3435			0.9978	0.1816	0.0970	-0.0365	-0.0022	0.0343	0.0094	
200	0.3505			0.9980	0.1542	0.0788	-0.0295	-0.0020	0.0246	0.0062	
250	0.3460			0.9937	0.1481	0.0752	-0.0340	-0.0063	0.0231	0.0057	
300	0.3558			0.9959	0.1274	0.0647	-0.0242	-0.0041	0.0168	0.0042	
0.63	8.00			10	0.3733	7.6350	0.3499	2.9069	-0.2567	-0.3650	0.1882
		25	0.4661	7.7597	0.2916	1.9955	-0.1639	-0.2403	0.1118	4.0358	
		50	0.5388	7.8766	0.2315	1.4962	-0.0912	-0.1234	0.0619	2.2517	
		75	0.5694	7.9395	0.1913	1.3317	-0.0606	-0.0605	0.0402	1.7752	
		100	0.5756	7.9348	0.1700	1.1582	-0.0544	-0.0652	0.0318	1.3444	
		150	0.5976	7.9546	0.1310	0.9263	-0.0324	-0.0454	0.0182	0.8592	
		200	0.6120	8.0148	0.1082	0.8081	-0.0180	0.0148	0.0120	0.6527	
		250	0.6091	7.9559	0.0974	0.6988	-0.0209	-0.0441	0.0099	0.4897	
		300	0.6107	7.9764	0.0922	0.6610	-0.0193	-0.0236	0.0089	0.4370	
		0.94	0.30	10	0.6306	0.2287	0.3586	0.1799	-0.3094	-0.0713	0.2242
25	0.8171			0.2700	0.1979	0.1590	-0.1229	-0.0300	0.0542	0.0262	
50	0.8848			0.2843	0.1042	0.1186	-0.0552	-0.0157	0.0139	0.0143	
75	0.9035			0.2874	0.0730	0.1012	-0.0365	-0.0126	0.0067	0.0104	
100	0.9171			0.2964	0.0544	0.0903	-0.0229	-0.0036	0.0035	0.0082	
150	0.9232			0.2955	0.0421	0.0768	-0.0168	-0.0045	0.0021	0.0059	
200	0.9271			0.2957	0.0342	0.0662	-0.0129	-0.0043	0.0013	0.0044	
250	0.9297			0.2968	0.0307	0.0589	-0.0103	-0.0032	0.0010	0.0035	
300	0.9322			0.2991	0.0282	0.0564	-0.0078	-0.0009	0.0009	0.0032	
0.08	1.70			10	0.1766	1.7486	0.2704	0.3914	0.0966	0.0486	0.0824
		25	0.1635	1.7483	0.2307	0.2588	0.0835	0.0483	0.0601	0.0693	
		50	0.1527	1.7566	0.2011	0.1888	0.0727	0.0566	0.0457	0.0388	
		75	0.1391	1.7420	0.1761	0.1554	0.0591	0.0420	0.0345	0.0259	
		100	0.1331	1.7349	0.1627	0.1301	0.0531	0.0349	0.0292	0.0181	
		150	0.1241	1.7336	0.1486	0.1164	0.0441	0.0336	0.0240	0.0147	
		200	0.1116	1.7247	0.1310	0.0964	0.0316	0.0247	0.0181	0.0099	
		250	0.1099	1.7208	0.1218	0.0884	0.0299	0.0208	0.0157	0.0082	
		300	0.0958	1.7133	0.1097	0.0755	0.0158	0.0133	0.0123	0.0059	

$$\frac{\partial \mathcal{L}(\Theta)}{\partial \sigma} = -\frac{2n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^n x_i^2 + \frac{2p}{\sigma^3} \sum_{i=1}^n \frac{x_i^2 \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}}{1 - p \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}}.$$

By setting the above partial derivatives to zero we obtain two non-linear equations whose solutions are the maximum likelihood estimates (MLEs) of the parameters, denoted by $\hat{\Theta}$ and must be obtained by certain numerical optimization procedure such as the Newton-Raphson iterative method. The observed Fisher information matrix denoted by $\mathbf{I}(\hat{\Theta}) = [\mathbf{I}_{\theta_i, \theta_j}]_{2 \times 2}$, $i, j = 1, 2$ can be obtained numerically in R software. The total observed Fisher information matrix can be approximated by

$$\mathbf{J}_n(\hat{\Theta}) \approx \left(-\frac{\partial^2 \mathcal{L}}{\partial \theta_i \partial \theta_j} \Big|_{\Theta = \hat{\Theta}} \right)_{2 \times 2}; \quad i, j = 1, 2. \tag{15}$$

For a given set of observations, the matrix in Equation (12) can be obtained after the convergence of the Newton-Raphson procedure via the `nlm` function in R software. The `nlm` function carry out a minimization of the function $\mathcal{L}(\Theta)$ using a Newton-type algorithm. Moreover, the elements of $\mathbf{J}_n(\hat{\Theta})$ are given by:

$$\frac{\partial^2 \mathcal{L}(\Theta)}{\partial p^2} = -\frac{n}{(1-p)^2} + 2 \sum_{i=1}^n \frac{\exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}}{\left(1 - p \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}\right)^2},$$

$$\frac{\partial^2 \mathcal{L}(\Theta)}{\partial p \partial \sigma} = \frac{2}{\sigma^3} \sum_{i=1}^n \frac{x_i^2 \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}}{1 - p \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}} + \frac{2p}{\sigma^3} \sum_{i=1}^n \frac{x_i^2 \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}}{\left(1 - p \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}\right)^2},$$

and

$$\begin{aligned} \frac{\partial^2 \mathcal{L}(\Theta)}{\partial \sigma^2} &= \frac{2n}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=1}^n x_i^2 - \frac{6p}{\sigma^4} \sum_{i=1}^n \frac{x_i^2 \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}}{1 - p \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}} \\ &+ \frac{2p}{\sigma^6} \sum_{i=1}^n \frac{x_i^4 \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}}{1 - p \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}} \end{aligned}$$

$$+ \frac{2p^2}{\sigma^6} \sum_{i=1}^n \frac{x_i^4 \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}}{\left(1 - p \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}\right)^2}.$$

4.1. Simulation study

In this section, we study the performance of the method of maximum likelihood estimation in estimating the parameters of the RGD using Monte-Carlo simulation for different sample sizes n and different parameter values. The simulation study was implemented in R software and it involves 10,000 replications for different sample sizes $n = 10, 25, 50, 75, 100, 150, 200, 250,$ and 300 and different parameter combinations. Table 3 gives the values of the mean estimates, mean standard errors (SEs), mean biases, and mean square errors (MSEs) of the parameters, p and σ for different sample sizes. It is easy to verify from these results that the MLEs approximates the true parameter values as n increases. Moreover, the SEs, biases, and MSEs decrease with increasing n . Finally, the MLE procedure can be used to estimate the parameters of the RGD effectively even for small n .

The Monte-Carlo algorithm is outlined as follows.

- (i) For specific parameter values $\Theta(p$ and $\sigma)$ simulate a random sample of size n from the RGD by Corollary 3.1.2.
- (ii) Estimate the parameters of the RGD by the method of MLE.
- (iii) Perform 10,000 replications of steps (i)-(ii).
- (iv) For each of the two parameters of the RGD compute the mean, standard error, bias and mean square error of the 10,000 parameter estimates. The analytical expressions of these statistics are given by $\bar{\hat{\Theta}} = \frac{1}{10,000} \sum_{i=1}^{10,000} \hat{\Theta}_i$, $SE_{\hat{\Theta}} = \sqrt{\frac{1}{10,000} \sum_{i=1}^{10,000} (\hat{\Theta}_i - \bar{\hat{\Theta}})^2}$, $Bias_{\hat{\Theta}} = \frac{1}{10,000} \sum_{i=1}^{10,000} (\hat{\Theta}_i - \Theta)$, and $MSE_{\hat{\Theta}} = \frac{1}{10,000} \sum_{i=1}^{10,000} (\hat{\Theta}_i - \Theta)^2$, respectively. Where $\hat{\Theta}_i$ is the MLE $(\hat{p}_i, \hat{\sigma}_i)'$ for each iteration and specific sample size n , $\bar{\hat{\Theta}} = (\bar{\hat{p}}, \bar{\hat{\sigma}})'$ corresponds to the mean of each of the parameters in $(\hat{p}_i, \hat{\sigma}_i)'$, and Θ is the actual values of the parameter vector $(p, \sigma)'$.

5. Analysis

In this section, we use a real data-set to demonstrate the performance of the Rayleigh-geometric distribution in modeling real-life data-set. The first data we consider is the yearly data on the carbon dioxide (CO₂) emissions in Afghanistan from 1980 to 2006 (1×10^6 metric tons). The total CO₂ emissions from the consumption and flaring of fossil fuels (in millions of metric tons of carbon dioxide) were compiled by the US Department of energy www.eia.doe.gov/emeu/iea/. The data is available in the R package called `asbio` (R Core Team, 2014). The second data is on the waiting time (minutes) before service of 100 bank customers that were reported in Merovci and Elbatal (2013). The summary statistics for the two data-sets are listed in Table 4. We compare the fitting performance of the RGD in modeling the CO₂ and the waiting time data with the following competing one-parameter, two-parameter, and three-parameter distributions:

1. Weibull distribution

$$F_X(x) = 1 - \exp\left\{-\left(\frac{x}{\beta}\right)^\alpha\right\}; x > 0, \alpha, \beta > 0,$$

2. Rayleigh distribution

$$F_X(x) = 1 - \exp(-x^2/[2\sigma^2]); x \geq 0, \sigma > 0,$$

3. gamma distribution

$$F_X(x) = \frac{\gamma\left(\alpha, \frac{x}{\beta}\right)}{\Gamma(\alpha)}; x > 0, \alpha, \beta > 0,$$

Table 4

Summary statistics for the CO₂ and waiting time data.

Statistics	CO ₂	Waiting time
	values	values
Minimum	0.680000	0.800000
Maximum	6.550000	38.500000
1st Quartile	1.350000	4.675000
3rd Quartile	2.155000	13.025000
Mean	2.149259	9.877000
Median	1.570000	8.100000
SE Mean	0.327301	0.723700
LCL Mean	1.476482	8.441023
UCL Mean	2.822037	11.312977
Variance	2.892407	52.374112
Stdev	1.700708	7.236996
Skewness	1.680579	1.450729
Kurtosis	1.626658	2.430040

4. exponentiated exponential (EE) distribution due to Gupta and Kundu (2001)

$$F_X(x) = [1 - \exp(-\lambda x)]^\alpha; x, \alpha, \lambda > 0,$$

5. Lindley distribution due to Lindley (1958)

$$F_X(x) = 1 - \left(1 + \frac{\theta x}{\theta + 1}\right) \exp(-\theta x); x > 0, \theta > 0$$

6. Lindley-geometric distribution (LGD) due to Zakerzadeh and Mahmoudi (2012)

$$F_X(x) = \frac{1 - \left(1 + \frac{\theta x}{\theta + 1}\right) \exp(-\theta x)}{1 - p \left(1 + \frac{\theta x}{\theta + 1}\right) \exp(-\theta x)}; x > 0, \theta > 0, p \in (0, 1)$$

7. transmuted Lindley-geometric distribution (TLGD) due to Merovci and Elbatal (2013)

$$F_X(x) = \frac{1 - \left(1 + \frac{\theta x}{\theta + 1}\right) \exp(-\theta x)}{1 - p \left(1 + \frac{\theta x}{\theta + 1}\right) \exp(-\theta x)} \times \left\{1 + \lambda - \lambda \left[\frac{1 - \left(1 + \frac{\theta x}{\theta + 1}\right) \exp(-\theta x)}{1 - p \left(1 + \frac{\theta x}{\theta + 1}\right) \exp(-\theta x)}\right]\right\};$$

$$x, \theta > 0, p \in (0, 1), |\lambda| \leq 1$$

8. exponential distribution

$$F_X(x) = 1 - \exp(-\lambda x); x \geq 0, \lambda > 0$$

9. EGD due to Adamidis and Loukas (1998)

$$F_X(x) = \frac{1 - \exp(-\beta x)}{1 - p \exp(-\beta x)}; x \geq 0, \beta > 0, p \in (0, 1).$$

10. Weibull-geometric distribution (WGD) due to Barreto-Souza et al. (2011)

$$F_X(x) = \frac{1 - \exp(-\beta x)^\alpha}{1 - p \exp(-\beta x)^\alpha}; x > 0, \alpha > 0, \beta > 0, p \in (0, 1).$$

The verification of the fitting performance for the fitted distributions would be carried out by comparing the values of their Kolmogorov-Smirnov K-S (see Kolmogorov, 1933, Smirnov, 1939, Scheffé, 1943; and Wolfowitz, 1949), log-likelihood ($-\mathcal{L}$), AIC, BIC, and the AICc goodness-of-fit statistics. The distribution with the smallest goodness-of-fit statistics and largest K-S p -value is the best. The analytical expressions for the goodness-of-fit measures are:

i. Kolmogorov-Smirnov K-S criterion

$$K-S = \max\left\{\frac{i}{n} - \hat{F}(x_{(i)}), \hat{F}(x_{(i)}) - \frac{i-1}{n}\right\},$$

Table 5
Parameter estimates, standard errors (in bracket), and the log-likelihood of the fitted models for the CO₂ data.

Models	MLEs		$-\widehat{\mathcal{L}}(\Theta)$	
	Parameters (SEs)			
RGD	$\hat{\rho} : 0.885253$ [0.1118244]	$\hat{\sigma} : 3.617573$ [1.5553301]	41.78893	
Weibull	$\hat{\alpha} : 1.464907$ [0.2003798]	$\hat{\beta} : 2.403237$ [0.3361784]	44.53341	
Rayleigh	$\hat{\sigma} : 1.924135$ [0.1851961]		47.63236	
Gamma	$\hat{\alpha} : 2.422926$ [0.6197570]	$\hat{\beta} : 1.127330$ [0.3203217]	42.91098	
EE	$\hat{\alpha} : 3.2244836$ [1.1167190]	$\hat{\lambda} : 0.9007314$ [0.1911419]	42.10793	
Lindley	$\hat{\theta} : 0.733655$ [0.1046472]		45.94749	
LGD	$\hat{\rho} : 2.458542 \times 10^{-6}$ [0.6827502]	$\hat{\theta} : 0.7366641$ [0.2483542]	45.94790	
TLGD	$\hat{\rho} : 0.03904947$ [0.8107832]	$\hat{\theta} : 1.16951716$ [0.3779581]	$\hat{\lambda} : -0.99999910$ [3.02056]	43.92790
Exp	$\hat{\lambda} : 0.4652761$ [0.08956167]		47.65833	
EGD	$\hat{\rho} : 1.345018 \times 10^{-6}$ [0.4269279]	$\hat{\beta} : 0.5186567$ [0.1689495]	47.82351	
WGD	$\hat{\rho} : 0.9887380$ [0.02168954]	$\hat{\alpha} : 2.8308259$ [0.46289655]	$\hat{\beta} : 0.1244236$ [0.08034716]	40.25773

Table 6
Goodness-of-fit of the fitted models for the CO₂ data.

Models	Information criteria			K-S	
	AIC	BIC	AICc	statistic	p-value
RGD	87.57786	90.16953	88.07786	0.1601	0.4936
Weibull	93.06682	95.65849	93.56682	0.1980	0.2402
Rayleigh	97.26473	98.56057	97.42473	0.3217	0.0075
Gamma	89.82196	92.41364	90.32196	0.1961	0.2504
EE	88.21587	90.80754	88.71587	0.1856	0.3099
Lindley	93.89498	95.19081	94.05498	0.2181	0.1534
LGD	95.89581	98.48748	96.39581	0.2192	0.1491
TLGD	93.85579	97.74330	94.89927	0.2000	0.2304
Exp	97.31666	98.61249	97.47666	0.2712	0.0377
EGD	99.64702	102.2387	100.1470	0.2972	0.0170
WGD	86.51550	90.40300	87.55890	0.1363	0.6972

ii. Akaike information criterion (AIC) due to Akaike (1974)

$$AIC = -2\widehat{\mathcal{L}} + 2k,$$

iii. Bayes information criterion (BIC) due to Schwarz (1978)

$$BIC = -2\widehat{\mathcal{L}} + k \log(n), \text{ and}$$

iv. AIC with a correction (AICc) due to Hurvich and Tsai (1989)

$$AICc = AIC + \frac{2k(k+1)}{n-k-1},$$

where, $\widehat{\mathcal{L}}$, k , n , and $\widehat{F}(\cdot)$ corresponds to the estimate of the model maximized log-likelihood function, number of parameters in the distribution, the sample size of the fitted data, and the estimated distribution function under the ordered data, respectively.

5.1. Discussion of results

Results from the model fittings are reported in Tables 5, 6, 7 and 8. For the CO₂ data, the RGD only gave the smallest BIC value while the WGD gave the smallest AIC, AICc, K-S statistic values, and the largest p -value for the K-S test in Table 6. For the waiting time data, the RGD gave the overall smallest information criteria values while the WGD gave the smallest K-S statistic value and largest p -value for the K-S test in Table 8. To avoid ambiguity, only the fits of the RGD and WGD are illustrated in Fig. 4 (a and b) and the plots reveal that the goodness-of-fit for the RGD and WGD are rather not questionable for the two data-sets. Moreover, the two-parameter RGD and three-parameter

Table 7
Parameter estimates, standard errors (in bracket), and the log-likelihood of the fitted models for the waiting time data.

Models	MLEs		$-\widehat{\mathcal{L}}(\Theta)$	
	Parameters (SEs)			
RGD	$\hat{\rho} : 0.8316597$ [0.08083768]	$\hat{\sigma} : 14.3553901$ [2.8466904]	316.9575	
Weibull	$\hat{\alpha} : 1.458487$ [0.1097911]	$\hat{\beta} : 10.955306$ [0.7944145]	318.7307	
Rayleigh	$\hat{\sigma} : 8.643073$ [0.4322616]		329.2400	
Gamma	$\hat{\alpha} : 2.0087855$ [0.26419876]	$\hat{\beta} : 0.2033796$ [0.03037547]	317.3001	
EE	$\hat{\alpha} : 2.1833594$ [0.3344291]	$\hat{\lambda} : 0.1591485$ [0.01750332]	317.0953	
Lindley	$\hat{\theta} : 0.1865713$ [0.01328208]		319.0374	
LGD	$\hat{\rho} : 2.571736 \times 10^{-7}$ [0.4213638]	$\hat{\theta} : 0.1863014$ [0.03360897]	319.0376	
TLGD	$\hat{\rho} : 1.123240 \times 10^{-7}$ [0.2287048]	$\hat{\theta} : 0.1885831$ [0.02248681]	$\hat{\lambda} : 0.04205799$ [NaN]	319.1569
Exp	$\hat{\lambda} : 0.1012454$ [0.01013453]		329.0209	
EGD	$\hat{\rho} : 6.299345 \times 10^{-6}$ [0.2588957]	$\hat{\beta} : 0.1017582$ [0.01862711]	329.0222	
WGD	$\hat{\rho} : 0.82143831$ [0.1575613]	$\hat{\alpha} : 1.97903008$ [0.2618073]	$\hat{\beta} : 0.05009027$ [0.01460418]	316.9542

Table 8
Goodness-of-fit of the fitted models for the waiting time data.

Models	Information criteria			K-S	
	AIC	BIC	AICc	statistic	p-value
RGD	637.9150	643.1254	638.0387	0.0391	0.9980
Weibull	641.4614	646.6717	641.5851	0.0578	0.8921
Rayleigh	660.4801	663.0853	660.5209	0.1736	0.0048
Gamma	638.6002	643.8106	638.7240	0.0425	0.9936
EE	638.1906	643.4009	638.3143	0.0402	0.9969
Lindley	640.0748	642.6800	640.1156	0.0677	0.7495
LGD	642.0752	647.2856	642.1989	0.0672	0.7571
TLGD	644.3137	652.1293	644.5637	0.0787	0.5661
Exp	660.0418	662.6469	660.0826	0.1730	0.0050
EGD	662.0445	667.2548	662.1682	0.1744	0.0046
WGD	639.9084	647.7239	640.1584	0.0380	0.9987

WGD are strong competitor models for the two data-sets because, the difference between the fit of the two distributions for the two data-sets is largely marginal (see Tables 6 and 8). However, in practice, the answer to the question of which between the two distributions (RGD and WGD) should be chosen is readily handy by the popular rule of parsimony which in principle favors the RGD distribution. To support our claim, we used the likelihood ratio test (LRT) to test the hypothesis $H_0 : \hat{\alpha} = 2$ and $\hat{\beta} = \frac{1}{\sigma} \sqrt{\frac{1}{2}}$ versus $H_1 : \hat{\alpha} \neq 2$ or $\hat{\beta} \neq \frac{1}{\sigma} \sqrt{\frac{1}{2}}$. The test statistic is $\Omega = 2[\widehat{\mathcal{L}}_{\text{Reduced model (RGD)}} - \widehat{\mathcal{L}}_{\text{Full model (WGD)}}]$ where $\widehat{\mathcal{L}}$ is the value of the estimated log-likelihood function and Ω follows the Chi-squared distribution with ν degrees of freedom i.e.; $\Omega \sim \chi^2_{\nu}$ where ν denote the difference between the number of parameters in the two distributions and the p -value for the test is calculated as $Pr(\chi^2_{\nu} > \Omega)$. The hypothesis is based on 0.05 level of significance and the non-rejection of H_0 suggests that the RGD is a better candidate for the fitted data than the WGD. The LRT result for the CO₂ data gave $\Omega = 3.0624$ and p -value = 0.0801232 and for the waiting time data $\Omega = 0.0066$ and p -value = 0.9352508 hence; there is no evidence at 0.05 level of significance to reject the H_0 (RGD) for the two data-sets and our claim is validated up to a reasonable extent.

A more recent paper by Anwar and Bibi (2018) introduced the three-parameter half-logistic generalized Weibull (HLGW) distribution with CDF

$$F_X(x) = \frac{1 - \exp(1 - [1 + \gamma x^{\eta}]^{\omega})}{1 + \exp(1 - [1 + \gamma x^{\eta}]^{\omega})}, \quad x > 0, \omega, \eta, \gamma > 0.$$

Anwar and Bibi (2018) applied the HLGW distribution to the waiting time data and the fit of the HLGW distribution was compared with those

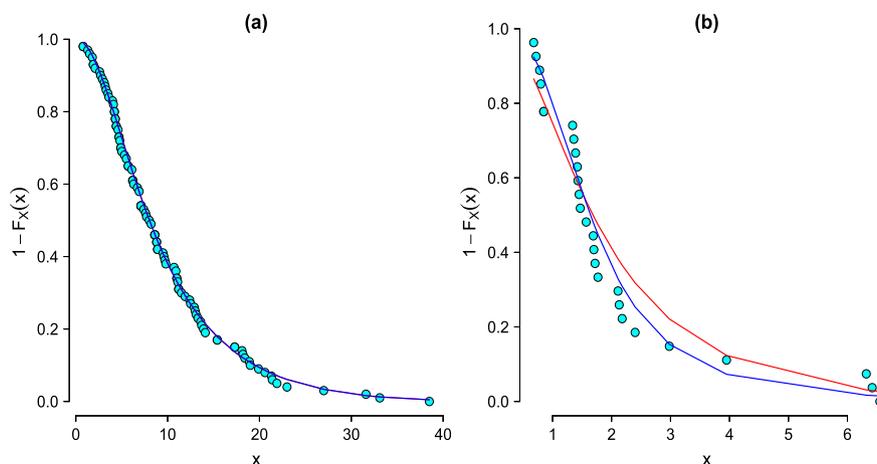


Fig. 4. Plots of the estimated reliability function for the RGD (red lines) and WGD (blue lines) superimposed on the empirical reliability function for the waiting time data (a) and CO₂ data (b).

of six competing one, two, and three-parameter distributions including the Weibull distribution. Based on the results from the distributions fittings in Table 4 of Anwar and Bibi (2018) the HLGW distribution indicate a better performance than the other distributions in modeling the waiting time data. Here, we want to point out that the two-parameter RGD in this paper offers a better fit to the waiting time data than the three-parameter HLGW distribution because the log-likelihood of the RGD (316.9575) is less than that of the HLGW distribution (317.1130). In terms of parsimony, the two-parameter RGD, gamma distribution, and EE distribution are generally better options for modeling the waiting time data than the HLGW distribution.

6. Conclusion

The statistical literature is lacking the complete representation of the exact parametrization of the Rayleigh-geometric distribution (RGD) as in this paper. On this note, we present a detailed account of the statistical properties of the two-parameter RGD with two applications to illustrate its possible utility. The RGD was obtained by mixing the geometric distribution and the Rayleigh distribution; moreover, the Rayleigh distribution is identified as the limiting case of the RGD when the mixing parameter p approaches zero. The hazard rate function of the distribution contains some important shapes of the hazard rate of some lifetime phenomena that are commonly encountered in practice, they are; increasing-decreasing-increasing and strictly increasing shapes. Estimation of the distribution parameters was carried out by the method of maximum likelihood and a numerical study show that the maximum likelihood estimation method provides good estimates for the distribution parameters. Two real-life examples based on the modeling of waiting time in bank queue and CO₂ emissions in Afghanistan were given to demonstrate the usefulness and fitting prowess of the RGD. We hope that this paper would encourage the application of the RGD in areas such as hydrology, reliability analysis, and meteorology.

Declarations

Author contribution statement

Idika E. Okorie: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Anthony C. Akpanta, Johnson Ohakwe, David C. Chikezie, Chris U. Onyemachi: Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

References

- Adamidis, K., Loukas, S., 1998. A lifetime distribution with decreasing failure rate. *Stat. Probab. Lett.* 39 (1), 35–42.
- Akaike, H., 1974. A new look at the statistical model identification. *IEEE Trans. Autom. Control* 19, 716–723.
- Anwar, M., Bibi, A., 2018. The half-logistic generalized Weibull distribution. *J. Probab. Stat.* 2018.
- Barreto-Souza, W., de Morais, A.L., Cordeiro, G.M., 2011. The Weibull-geometric distribution. *J. Stat. Comput. Simul.* 81 (5), 645–657.
- Galton, F., 1911. *Inquiries Into Human Faculty and Its Development*. JM Dent.
- Gupta, R.D., Kundu, D., 2001. Generalized exponential distribution: different method of estimations. *J. Stat. Comput. Simul.* 69 (4), 315–337.
- Hamedani, G.G., Ahsanullah, M., 2011. Characterizations of Weibull geometric distribution. *J. Stat. Theory Appl.* 10 (4), 581–590.
- Hurvich, C.M., Tsai, C.L., 1989. Regression and time series model selection in small samples. *Biometrika* 76, 297–307.
- Kilbas, A.A., Srivastava, H.M., Trujillo, J., 2006. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, vol. 204.
- Kolmogorov, A., 1933. Sulla determinazione empirica di una legge di distribuzione. *Inst. Ital. Attuari, Giorn.* 4, 83–91.
- Kuş, C., 2007. A new lifetime distribution. *Comput. Stat. Data Anal.* 51 (9), 4497–4509.
- Lindley, D.V., 1958. Fiducial distributions and Bayes' theorem. *J. R. Stat. Soc., Ser. B, Methodol.*, 102–107.
- Mahmoudi, E., Shiran, M., 2012. Exponentiated Weibull-geometric distribution and its applications. *arXiv preprint arXiv:1206.4008*.
- Marshall, A.W., Olkin, I., 1997. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika* 84 (3), 641–652.
- Mathai, A.M., Saxena, R.K., 1978. *The H-Function with Applications in Statistics and Other Disciplines*. John Wiley and Sons, New York.
- Merovci, F., Elbatal, I., 2013. Transmuted Lindley-geometric distribution and its applications. *arXiv preprint arXiv:1309.3774*.
- Moors, J.J.A., 1988. A quantile alternative for kurtosis. *Statistician*, 25–32.
- Mudholkar, G.S., Srivastava, D.K., 1993. Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Trans. Reliab.* 42 (2), 299–302.
- Nadarajah, S., Shahsanaei, F., Rezaei, S., 2014. A new four-parameter lifetime distribution. *J. Stat. Comput. Simul.* 84 (2), 248–263.
- Okorie, I.E., Akpanta, A.C., Ohakwe, J., Chikezie, D.C., 2017a. The modified Power function distribution. *Cogent Math.* 4 (1), 1319592.

- Okorie, I.E., Akpanta, A.C., Ohakwe, J., Chikezie, D.C., Obi, E.O., 2017b. The adjusted Fisk Weibull distribution: properties and applications. *Int. J. Model. Simul.*, 1–12.
- Ozel, G., Cakmakyapan, S., 2015. A new distribution and application on the lifetime data. *Int. J. Comput. Math. Sci.* 9 (7), 420–423.
- R Core Team, 2014. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria. URL <http://www.R-project.org/>.
- Ristić, M.M., Nadarajah, S., 2014. A new lifetime distribution. *J. Stat. Comput. Simul.* 84 (1), 135–150.
- Scheffé, H., 1943. Statistical inference in the non-parametric case. *Ann. Math. Stat.* 14 (4), 305–332.
- Schwarz, G.E., 1978. Estimating the dimension of a model. *Ann. Stat.* 6, 461–464.
- Smirnov, N., 1939. Sur les écarts de la courbe de distribution empirique. *Mat. Sb.* 48 (1), 3–26.
- Srivastava, H.M., Gupta, K.C., Goyal, S.P., 1982. *The H-Functions of One and Two Variables with Applications*. South Asian Publ., New Delhi-Madras.
- Tahir, M.H., Zubair, M., Cordeiro, G.M., Alzaatreh, A., Mansoor, M., 2016. The Poisson-X family of distributions. *J. Stat. Comput. Simul.* 86 (14), 2901–2921.
- Wolfowitz, J., 1949. Non-parametric statistical inference. In: *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability*. University of California Press, Berkeley, CA, pp. 93–113.
- Wright, E.M., 1935. The asymptotic expansion of the generalized hypergeometric function. *J. Lond. Math. Soc.* 1 (4), 286–293.
- Zakerzadeh, H., Mahmoudi, E., 2012. A new two parameter lifetime distribution: model and properties. arXiv preprint arXiv:1204.4248.