



On some generated soft topological spaces and soft homogeneity

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ABSTRACT

We introduce soft homogeneity as an extension of homogeneity in ordinary topological spaces. Based on the generated soft topology of a given indexed family of classical topologies inspite of a one topology given by Terepeta in [16], we investigate soft minimal open set and homogeneity relation between the generated soft topology and the given indexed family of topologies. We introduce several results, examples and counterexamples.

1. Introduction

The classical mathematical theories have difficulties for solving complicated problems which include uncertain data in many areas such as engineering, environment, economics, medical science, social science, etc. Theories of probability, fuzzy sets [1], rough sets [2], intuitionistic fuzzy sets [3] and vague sets [4] are considered as mathematical tools for dealing with uncertainties. Molodtsov [5] justified that each of these theories has its deep-seated difficulty. These difficulties are mainly come from the inadequacy of the parameterization tool of the theories. For dealing with uncertainties away from these difficulties, Molodtsov [5] defined soft sets as follows: Let X be an initial universe and A be a set of parameters and denote the power set of X by $\mathcal{P}(X)$, a soft set over X relative to A is a function $F : A \rightarrow \mathcal{P}(X)$. The theory of soft sets has been introduced and studied by several researchers (see [6, 7]). Authors [5, 8] applied soft sets in many areas such as Riemann integration, Perron integration, smoothness of function, operation research, game theory, probability and theory of measurements. Authors [9] applied soft sets in decision-making problems. The notion of soft topological spaces was introduced in [10]. Then, Mathematicians modified several concepts of classical topological spaces to include soft topological spaces (see [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]). A topological space (X, \mathfrak{S}) is homogeneous if for all $x, y \in X$, there is a homeomorphism $f : (X, \mathfrak{S}) \rightarrow (X, \mathfrak{S})$ such that $f(x) = y$. Since homogeneity concepts are of importance in general topology and still a hot area of research, as appears in [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33], we see that it is suitable to extend the homogeneity concept to include soft topological spaces. One of our main goals of the present work is to show how the definition of ho-

mogeneity in ordinary topological spaces can be modified in order to define its extension in soft topological spaces. This paper is organized as follows.

In section two, we introduce some basic definitions and results which we use them in our research.

In section three, for a given collection $\{\mathfrak{S}_a : a \in A\}$ of topologies on a set X , we introduce a generated soft topology τ over X with a set of parameters A such that $\tau_a = \mathfrak{S}_a$ for all $a \in A$; moreover, for a given topological space (X, \mathfrak{S}) and a given set of parameters A , we show that (X, \mathfrak{S}) generates a soft topology τ on X with the set of parameters A .

In section four, we continue the study of minimal soft open sets; in particular, we give a mapping regarding minimal soft open sets, also we give a link between the minimal soft open sets of a soft topological space and the minimal open sets of its generated topological spaces and vice versa.

In section five, we introduce and investigate soft homogeneity in soft topological spaces; in particular, we study the relation between soft homogeneity of a soft topological space and some ordinary topological spaces generated by this soft topological space, we study the relation between homogeneity of an ordinary topological space and some soft topological spaces generated by this ordinary topological space, and we introduce some properties of homogeneous soft topological spaces that contains a minimal soft open set.

2. Preliminaries

In this section, we introduce some basic definitions and results which we use them in our research.

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Definition 2.1. [5] Let X be an initial universe and A be a set of parameters. A soft set over X relative to A is a function $F : A \rightarrow \mathcal{P}(X)$. The family of all soft sets over X relative to A will be denoted by $SS(X, A)$.

Definition 2.2. [6] Let $F, G \in SS(X, A)$.

(1) F is a soft subset of G , denoted by $F \subseteq G$, if $F(a) \subseteq G(a)$ for each $a \in A$.

(2) F and G are said to be soft equal, denoted by $F = G$ if $F \subseteq G$ and $G \subseteq F$.

(3) Union of F and G is denoted by $F \cup G$ and defined to be the soft set $F \cup G \in SS(X, A)$ where $(F \cup G)(a) = F(a) \cup G(a)$ for each $a \in A$.

(4) Intersection of F and G is denoted by $F \cap G$ and defined to be the soft set $F \cap G \in SS(X, A)$ where $(F \cap G)(a) = F(a) \cap G(a)$ for each $a \in A$.

Definition 2.3. [11] Let Δ be an arbitrary index set and $\{F_\alpha : \alpha \in \Delta\} \subseteq SS(X, A)$.

(a) The union of these soft sets is the soft set denoted by $\bigcup_{\alpha \in \Delta} F_\alpha$ and defined by $(\bigcup_{\alpha \in \Delta} F_\alpha)(a) = \bigcup_{\alpha \in \Delta} F_\alpha(a)$ for each $a \in A$.

(b) The intersection of these soft sets is the soft set denoted by $\bigcap_{\alpha \in \Delta} F_\alpha$ and defined by $(\bigcap_{\alpha \in \Delta} F_\alpha)(a) = \bigcap_{\alpha \in \Delta} F_\alpha(a)$ for each $a \in A$.

Definition 2.4. [6] Let $F \in SS(X, A)$.

(a) F is called a null soft set over X relative to A , denoted by 0_A , if $F(a) = \emptyset$ for each $a \in A$.

(b) F is called an absolute soft set over X relative to A , denoted by 1_A , if $F(a) = X$ for each $a \in A$.

Definition 2.5. [12] Let $F \in SS(X, A)$. F is called a soft point over X relative to A if there exist $e \in A$ and $x \in X$ such that

$$F(a) = \begin{cases} \{x\} & \text{if } a = e \\ \emptyset & \text{if } a \neq e \end{cases}$$

We denote F by e_x . The family of all soft points over X relative to A is denoted by $SP(X, A)$.

Proposition 2.6. [12] Let $a_x, b_y \in SP(X, A)$. Then $a_x = b_y$ iff $x = y$ and $a = b$.

Definition 2.7. [12] Let $F \in SS(X, A)$ and $e_x \in SP(X, A)$. Then e_x is said to belong to F (notation: $e_x \in F$) if $e_x \subseteq F$ or equivalently: $e_x \in F$ iff $x \in F(e)$.

Proposition 2.8. $F \in SS(X, A) - \{0_A\}$ iff there is $a_x \in SP(X, A)$ such that $a_x \in F$.

Proof. Straightforward. \square

Proposition 2.9. [12] Let $F, G \in SS(X, A)$. Then the following are equivalent:

- (a) $F \subseteq G$.
- (b) For all $a_x \in SP(X, A)$, $a_x \in F$ implies $a_x \in G$.

Definition 2.10. [12] Let $F \in SS(X, A)$. The set $\{a_x : a_x \in F\}$ will be denoted by $Pt(F)$.

It is clear that $SP(X, A) = Pt(1_A)$.

Proposition 2.11. [12] Let X be an initial universe and A be a set of parameters. Then for any $F, G \in SS(X, A)$, $F \subseteq G$ iff $Pt(F) \subseteq Pt(G)$.

Definition 2.12. [7] Let $SS(X, A)$ and $SS(Y, B)$ be families of soft sets. Let $p : X \rightarrow Y$ and $u : A \rightarrow B$ be functions. Then a soft mapping $f_{pu} : SS(X, A) \rightarrow SS(Y, B)$ is defined as:

(a) Let $F \in SS(X, A)$. The image of F under f_{pu} , written as $f_{pu}(F) \in SS(Y, B)$ is defined by

$$(f_{pu}(F))(b) = \begin{cases} \cup \{p(F(a)) : u(a) = b\} & \text{if } u^{-1}(b) \neq \emptyset \\ \emptyset & \text{if } u^{-1}(b) = \emptyset. \end{cases}$$

(b) Let $G \in SS(Y, B)$. The inverse of G under f_{pu} , written as $f_{pu}^{-1}(G) \in SS(X, A)$ is defined by

$$(f_{pu}^{-1}(G))(a) = p^{-1}(G(u(a))).$$

Definition 2.13. [7] Let $f_{pu} : SS(X, A) \rightarrow SS(Y, B)$ be a soft mapping. Then f_{pu} is called:

- (a) Injective if p and u are injective.
- (b) Surjective if p and u are surjective.
- (c) Bijective if p and u are bijective.

Proposition 2.14. Let $f_{pu} : SS(X, A) \rightarrow SS(Y, B)$ be a soft mapping. Then for all $a_x \in SP(X, A)$, $f_{pu}(a_x) = u(a)_{p(x)} \in SP(Y, B)$.

Proof. Straightforward. \square

Proposition 2.15. Let $f_{pu} : SS(X, A) \rightarrow SS(Y, B)$ be a soft mapping. Then for all $a_x \in SP(X, A)$, $f_{pu}(a_x) = b_y$ iff $p(x) = y$ and $u(a) = b$.

Proof. Straightforward. \square

Proposition 2.16. A soft mapping $f_{pu} : SS(X, A) \rightarrow SS(Y, B)$ is injective iff $f_{pu} : SP(X, A) \rightarrow SP(Y, B)$ is injective.

Proof. Straightforward. \square

Proposition 2.17. A soft mapping $f_{pu} : SS(X, A) \rightarrow SS(Y, B)$ is surjective iff $f_{pu} : SP(X, A) \rightarrow SP(Y, B)$ is surjective.

Proof. Straightforward. \square

Proposition 2.18. Let $f_{pu} : SS(X, A) \rightarrow SS(Y, B)$ be a soft mapping. Let $b_y \in SP(Y, B)$ and $F \in SS(X, A)$. Then $b_y \in f_{pu}(F)$ iff there is $a_x \in F$ such that $f_{pu}(a_x) = b_y$.

Proof. Necessity. Assume that $b_y \in f_{pu}(F)$. Then $y \in (f_{pu}(F))(b) = \cup \{p(F(a)) : a \in u^{-1}(b)\}$. So, there is $a \in u^{-1}(b)$ such that $y \in p(F(a))$. Choose $x \in F(a)$ such that $p(x) = y$. Thus, we have $a_x \in F$ and $f_{pu}(a_x) = u(a)_{p(x)} = b_y$.

Sufficiency. Assume that there is $a_x \in F$ such that $f_{pu}(a_x) = b_y$. Since $b_y \in F$, then $f_{pu}(a_x) \in f_{pu}(F)$ and so $b_y \in f_{pu}(F)$. \square

Proposition 2.19. Let $f_{pu} : SS(X, A) \rightarrow SS(Y, B)$ be a soft mapping. Let $a_x \in SP(X, A)$ and $G \in SS(Y, B)$. Then $a_x \in f_{pu}^{-1}(G)$ iff $f_{pu}(a_x) \in G$.

Proof. Necessity. Assume that $a_x \in f_{pu}^{-1}(G)$. Then $x \in (f_{pu}^{-1}(G))(a) = p^{-1}(G(u(a)))$. So, $p(x) \in G(u(a))$. Thus, $f_{pu}(a_x) = u(a)_{p(x)} \in G$.

Sufficiency. Assume that $f_{pu}(a_x) \in G$. Then $u(a)_{p(x)} \in G$ and $p(x) \in G(u(a))$. Thus, $x \in p^{-1}(G(u(a))) = (f_{pu}^{-1}(G))(a)$. Hence, $a_x \in f_{pu}^{-1}(G)$. \square

Definition 2.20. [10] Let $\tau \subseteq S(X, A)$. Then τ is called a soft topology on X relative to A if

- (1) $0_A, 1_A \in \tau$,
- (2) the union of any number of soft sets in τ belongs to τ ,
- (3) the intersection of any two soft sets in τ belongs to τ .

The triplet (X, τ, A) is called a soft topological space (STS) over X relative to A . The members of τ are called soft open sets in (X, τ, A) and their complements are called soft closed sets in (X, τ, A) .

Definition 2.21. A soft mapping $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ is called:

- (1) Soft continuous [15] if $f_{pu}^{-1}(G) \in \tau$ for every $G \in \sigma$.
- (2) Soft open [15] if $f_{pu}(F) \in \sigma$ for every $F \in \tau$.
- (3) A soft homeomorphism [14] if f_{pu} is soft continuous, soft open and bijective.

Definition 2.22. [15] Let $f_{p_1u_1} : SS(X, A) \rightarrow SS(Y, B)$ and $f_{p_2u_2} : SS(Y, B) \rightarrow SS(Z, C)$ be functions, then the composition of $f_{p_1u_1}$ and $f_{p_2u_2}$ is soft mapping from $SS(X, A)$ onto $SS(Z, C)$ denoted by $f_{p_1u_1} \circ f_{p_2u_2}$ and defined by $f_{p_1u_1} \circ f_{p_2u_2} = f_{(p_2 \circ p_1)(u_2 \circ u_1)}$.

Proposition 2.23. [15] If $f_{p_1u_1} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ and $f_{p_2u_2} : (Y, \sigma, B) \rightarrow (Z, \beta, C)$ are soft continuous, then $f_{(p_2 \circ p_1)(u_2 \circ u_1)} : (X, \tau, A) \rightarrow (Z, \beta, C)$ is soft continuous.

Proposition 2.24. [13] A soft mapping $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ is a soft homeomorphism iff f_{pu} is bijective, and $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ and $f_{p^{-1}u^{-1}} : (Y, \sigma, B) \rightarrow (X, \tau, A)$ are soft continuous.

Proposition 2.25. Let (X, τ, A) be a STS. Then the set of all soft homeomorphisms from (X, τ, A) onto (X, τ, A) forms a group under the operation $(f_{p_2u_2}) \circ (f_{p_1u_1}) = f_{(p_2 \circ p_1)(u_2 \circ u_1)}$.

Proof. Straightforward. \square

Proposition 2.26. [10] Let (X, τ, A) be a STS. Then the collection $\{F(a) : F \in \tau\}$ defines a topology on X for every $a \in A$. This topology will be denoted by τ_a .

Definition 2.27. [28] Let (X, \mathfrak{S}) be a topological space. A non-empty open subset $A \subseteq X$ is called a minimal open set if the only open subsets of A are A and \emptyset . The set of all minimal open sets of (X, \mathfrak{S}) will be denoted by $\min(X, \mathfrak{S})$.

Definition 2.28. [19] Let (X, τ, A) be a soft topological space. A sub-collection B of τ is called a soft base of τ if every member of τ can be expressed as a union of members of B .

Proposition 2.29. [20] Let (X, τ, A) be a STS and let $B \subseteq \tau$. Then B is a soft base for τ if for every $F \in \tau - \{0_A\}$ and every $a_x \in F$, there exists $G \in B$ such that $a_x \in G \subseteq F$.

Proposition 2.30. [17] A collection $B \subseteq SS(X, A)$ is a soft base for some soft topology on X relative to A iff the following conditions hold:

- i. $\bigcup \{F : F \in B\} = 1_A$.
- ii. For every $F, G \in B$ and for every $a_x \in F \cap G$, there is $H \in B$ such that $a_x \in H \subseteq F \cap G$.

3. Soft topologies generated by ordinary topologies

In this section, for a given collection $\{\mathfrak{S}_a : a \in A\}$ of topologies on a set X , we introduce a generated soft topology τ over X with a set of parameters A such that $\tau_a = \mathfrak{S}_a$ for all $a \in A$; moreover, for a given topological space (X, \mathfrak{S}) and a given set of parameters A we show that (X, \mathfrak{S}) generates a soft topology τ on X with the set of parameters A .

Theorem 3.1. Let X be an initial universe and let A be a set of parameters. Let $\{\mathfrak{S}_a : a \in A\}$ be an indexed family of topologies on X and let

$$\tau = \{F \in SS(X, A) : F(a) \in \mathfrak{S}_a \text{ for all } a \in A\}.$$

Then (X, τ, A) is a STS.

Proof. For all $a \in A$, $\emptyset, X \in \mathfrak{S}_a$ and so $0_A, 1_A \in \tau$. Let $F, G \in \tau$, then for all $a \in A$, $F(a), G(a) \in \mathfrak{S}_a$ and $F(a) \cap G(a) \in \mathfrak{S}_a$. So for all $a \in A$, $F(a) \cap G(a) = (F \cap G)(a) \in \mathfrak{S}_a$ and hence $F \cap G \in \tau$. Let $\{F_\alpha : \alpha \in \Delta\} \subseteq \tau$, then for all $a \in A$ and $\alpha \in \Delta$, $F_\alpha(a) \in \mathfrak{S}_a$ and $\left(\bigcup_{\alpha \in \Delta} F_\alpha(a)\right) \in \mathfrak{S}_a$. So for all $a \in A$, $\left(\bigcup_{\alpha \in \Delta} F_\alpha(a)\right) = \left(\bigcup_{\alpha \in \Delta} F_\alpha\right)(a) \in \mathfrak{S}_a$ and hence $\bigcup_{\alpha \in \Delta} F_\alpha \in \tau$. \square

Definition 3.2. Let X be an initial universe and let A be a set of parameters. Let $\{\mathfrak{S}_a : a \in A\}$ be an indexed family of topologies on X . Then the soft topology $\{F \in SS(X, A) : F(a) \in \mathfrak{S}_a \text{ for all } a \in A\}$ will be denoted by $\bigoplus_{a \in A} \mathfrak{S}_a$.

For any topological space (X, \mathfrak{S}) and any set of parameters A , denote the family $\{F \in SS(X, A) : F(a) \in \mathfrak{S} \text{ for all } a \in A\}$ by $\tau(\mathfrak{S})$.

Corollary 3.3. [16] For any topological space (X, \mathfrak{S}) and any set of parameters A , $(X, \tau(\mathfrak{S}), A)$ is a STS.

Proof. For each $a \in A$, set $\mathfrak{S}_a = \mathfrak{S}$. Then $\tau(\mathfrak{S}) = \bigoplus_{a \in A} \mathfrak{S}_a$ and by Theorem 3.1 we get the result. \square

Definition 3.4. Let X be an initial universe and let A be a set of parameters. For any $a \in A$ and $Y \subseteq X$, the soft set $F \in SS(X, A)$ defined by

$$F(b) = \begin{cases} Y & \text{if } b = a \\ \emptyset & \text{if } b \neq a \end{cases}$$

will be denoted by a_Y .

Theorem 3.5. Let X be an initial universe and let A be a set of parameters. Let $\{\mathfrak{S}_a : a \in A\}$ be an indexed family of topologies on X . Then the family $\{a_Y : a \in A \text{ and } Y \in \mathfrak{S}_a\}$ is a soft base of $\bigoplus_{a \in A} \mathfrak{S}_a$.

Proof. If $F \in \bigoplus_{a \in A} \mathfrak{S}_a - \{0_A\}$, then $F = \bigcap_{a \in A} a_{F(a)}$. \square

Corollary 3.6. Let X be an initial universe and let A be a set of parameters. Let $\{\mathfrak{S}_a : a \in A\}$ be an indexed family of topologies on X . If B_a is a base for \mathfrak{S}_a for all $a \in A$, then $\{a_Y : a \in A \text{ and } Y \in B_a\}$ is a soft base of $\bigoplus_{a \in A} \mathfrak{S}_a$.

Theorem 3.7. Let X be an initial universe and let A be a set of parameters. Let $\{\mathfrak{S}_a : a \in A\}$ be an indexed family of topologies on X . Then $\left(\bigoplus_{a \in A} \mathfrak{S}_a\right)_b = \mathfrak{S}_b$ for all $b \in A$.

Proof. Let $S \in \left(\bigoplus_{a \in A} \mathfrak{S}_a\right)_b$, then there is $F \in \bigoplus_{a \in A} \mathfrak{S}_a$ such that $F(b) = S$. By the definition of $\bigoplus_{a \in A} \mathfrak{S}_a$, $F(b) \in \mathfrak{S}_b$ and thus $S \in \mathfrak{S}_b$. Conversely, if $S \in \mathfrak{S}_b$, then $b_S \in \bigoplus_{a \in A} \mathfrak{S}_a$ and so $b_S(b) = S \in \left(\bigoplus_{a \in A} \mathfrak{S}_a\right)_b$. \square

Corollary 3.8. [16] If (X, \mathfrak{S}) is a topological space and A is any set of parameters, then $(\tau(\mathfrak{S}))_a = \mathfrak{S}$ for all $a \in A$.

Proof. For each $a \in A$, set $\mathfrak{S}_a = \mathfrak{S}$. Then $\tau(\mathfrak{S}) = \bigoplus_{a \in A} \mathfrak{S}_a$ and by Theorem 3.7 we get the result. \square

Theorem 3.9. If (X, τ, A) is a STS, then $\tau \subseteq \bigoplus_{a \in A} \tau_a$.

Proof. Let $F \in \tau$, then $F(a) \in \tau_a$ for all $a \in A$ and so $F \in \bigoplus_{a \in A} \tau_a$. \square

The following example shows that the equality in Theorem 3.9 does not hold in general:

Example 3.10. Let $X = \{1, 2, 3, 4\}$, $A = \{a, b\}$, $F = \{(a, \{1, 2\}), (b, \{3, 4\})\}$, and $\tau = \{0_A, 1_A, F\}$. Then $\tau_a = \{\emptyset, X, \{1, 2\}\}$, $\tau_b = \{\emptyset, X, \{3, 4\}\}$ and $\tau_a \oplus \tau_b = \{0_A, 1_A, a_{\{1,2\}}, b_{\{3,4\}}, F\}$. So $\tau \neq \tau_a \oplus \tau_b$.

Theorem 3.11. If (X, τ, A) is a STS, then $\left(\bigoplus_{a \in A} \tau_a\right)_b = \tau_b$ for all $b \in A$.

Proof. Theorem 3.7. \square

Definition 3.12. Let X be an initial universe and A be a set of parameters. A soft set $F \in SS(X, A)$ such that $F(a) = Y$ for all $a \in A$ will be called a constant soft set and will be denoted by C_Y . The family of all constant soft sets in $SS(X, A)$ will be denoted by $CSS(X, A)$.

Remark 3.13. Let X be an initial universe and A be a set of parameters. Then $0_A = C_\emptyset$ and $1_A = C_X$.

Lemma 3.14. Let $\{C_{Y_\alpha} : \alpha \in \Delta\} \subseteq CSS(X, A)$. Then $\widetilde{\bigcup_{\alpha \in \Delta} C_{Y_\alpha}} = C\left(\bigcup_{\alpha \in \Delta} Y_\alpha\right)$ and $\widetilde{\bigcap_{\alpha \in \Delta} C_{Y_\alpha}} = C\left(\bigcap_{\alpha \in \Delta} Y_\alpha\right)$.

Proof. Straightforward. \square

Theorem 3.15. If (X, \mathfrak{S}) is a topological space, then for any set of parameters A , the collection $\tau = \{C_Y : Y \in \mathfrak{S}\}$ is a soft topology on X relative to A .

Proof. Suppose that (X, \mathfrak{S}) is a topological space. Since $\emptyset, X \in \mathfrak{S}$, then $C_\emptyset, C_X \in \tau$ and so $0_A, 1_A \in \tau$. Let $C_Y, C_Z \in \tau$ where $Y, Z \in \mathfrak{S}$. Then $Y \cap Z \in \mathfrak{S}$ and so $C_{Y \cap Z} \in \tau$. By Lemma 3.14, $C_Y \widetilde{\cap} C_Z = C_{Y \cap Z}$ and thus $C_Y \widetilde{\cap} C_Z \in \tau$. Let $\{C_{Y_\alpha} : \alpha \in \Delta\} \subseteq \tau$ where $\{Y_\alpha : \alpha \in \Delta\} \subseteq \mathfrak{S}$, then $\bigcup_{\alpha \in \Delta} Y_\alpha \in \mathfrak{S}$ and so $C\left(\bigcup_{\alpha \in \Delta} Y_\alpha\right) \in \tau$. By Lemma 3.14, $\widetilde{\bigcup_{\alpha \in \Delta} C_{Y_\alpha}} = C\left(\bigcup_{\alpha \in \Delta} Y_\alpha\right)$ and thus $\widetilde{\bigcup_{\alpha \in \Delta} C_{Y_\alpha}} \in \tau$. This ends the proof that (X, τ, A) is a STS. \square

Definition 3.16. If (X, \mathfrak{S}) is a topological space and A is a set of parameters, then we will denote the soft topology $\{C_Y : Y \in \mathfrak{S}\}$ by $C(\mathfrak{S})$.

Theorem 3.17. If (X, \mathfrak{S}) is a topological space and A is a set of parameters, then $(C(\mathfrak{S}))_a = \mathfrak{S}$ for all $a \in A$.

Proof. Clear. \square

Theorem 3.18. Let (X, τ, A) be a STS with $\tau \subseteq CSS(X, A)$, then $\{Y \subseteq X : C_Y \in \tau\}$ is a topology on X .

Proof. Suppose that (X, τ, A) is a STS with $\tau \subseteq CSS(X, A)$. Let $\mathfrak{S} = \{Y \subseteq X : C_Y \in \tau\}$. Since $C_\emptyset, C_X \in \tau$, then $\emptyset, X \in \mathfrak{S}$. Let $Y, Z \in \mathfrak{S}$. Then $C_Y, C_Z \in \tau$ and so $C_Y \widetilde{\cap} C_Z \in \tau$. By Lemma 3.14, $C_Y \widetilde{\cap} C_Z = C_{Y \cap Z}$. Thus, $Y \cap Z \in \mathfrak{S}$. Let $\{Y_\alpha : \alpha \in \Delta\} \subseteq \mathfrak{S}$. Then $\{C_{Y_\alpha} : \alpha \in \Delta\} \subseteq \tau$ and so $\widetilde{\bigcup_{\alpha \in \Delta} C_{Y_\alpha}} \in \tau$. By Lemma 3.14, $\widetilde{\bigcup_{\alpha \in \Delta} C_{Y_\alpha}} = C\left(\bigcup_{\alpha \in \Delta} Y_\alpha\right)$. Thus, $\bigcup_{\alpha \in \Delta} Y_\alpha \in \mathfrak{S}$. This ends the proof that (X, \mathfrak{S}) is a topological space. \square

Definition 3.19. If (X, τ, A) is a STS with $\tau \subseteq CSS(X, A)$, then the topology $\{Y \subseteq X : C_Y \in \tau\}$ on X will be denoted by $D(\tau)$.

Theorem 3.20. If (X, τ, A) is a STS with $\tau \subseteq CSS(X, A)$, then $\tau_a = D(\tau)$ for all $a \in A$.

Remark 3.21. Let (X, \mathfrak{S}) be a topological space and let A be a set of parameters. Then $D(C(\mathfrak{S})) = \mathfrak{S}$.

Remark 3.22. Let (X, τ, A) be a STS with $\tau \subseteq CSS(X, A)$. Then $C(D(\tau)) = \tau$.

4. Minimal soft open sets

In this section, we continue the study of minimal soft open sets; in particular, we give a mapping regarding minimal soft open sets, also we give a link between the soft minimal open sets of a soft topological space and the minimal open sets of its generated topological spaces and vice versa.

Definition 4.1. [21] Let (X, τ, A) be a STS. A soft set $F \in SS(X, A)$ is said to be a minimal soft open set of (X, τ, A) if $F \in \tau - \{0_A\}$ and for all $G \in \tau$ with $G \widetilde{\subseteq} F$ either $G = 0_A$ or $G = F$.

The set of all minimal soft open sets of a STS (X, τ, A) will be denoted by $\min(X, \tau, A)$.

Proposition 4.2. [21] Let (X, τ, A) be a STS. If $F \in \min(X, \tau, A)$ and $G \in \tau$, then $F \widetilde{\cap} G = 0_A$ or $F \widetilde{\subseteq} G$.

Proposition 4.3. [21] Let (X, τ, A) be a STS. If $F, G \in \min(X, \tau, A)$, then $F \widetilde{\cap} G = 0_A$ or $F = G$.

Theorem 4.4. Let $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ be a soft continuous function. If $F \in \min(X, \tau, A)$ with $f_{pu}(F) \in \sigma$, then $f_{pu}(F) \in \min(Y, \sigma, B)$.

Proof. Since $F \in \min(X, \tau, A)$, then there is $a_x \widetilde{\in} F$. So, $f_{pu}(a_x) = u(a)_{p(x)} \widetilde{\in} f_{pu}(F)$ and hence $f_{pu}(F) \neq 0_B$. Suppose $H \in \sigma - \{0_B\}$ with $H \widetilde{\subseteq} f_{pu}(F)$. Choose $b_y \widetilde{\in} H$. Then $b_y \widetilde{\in} f_{pu}(F)$. So, there is $a_x \widetilde{\in} F$ such that $b_y = f_{pu}(a_x)$. Since f_{pu} is soft continuous, then $f_{pu}^{-1}(H) \in \tau$. Since we have $a_x \widetilde{\in} F \widetilde{\cap} f_{pu}^{-1}(H)$, then $F \widetilde{\cap} f_{pu}^{-1}(H) \in \tau - \{0_A\}$. So by Proposition 4.2, $F \widetilde{\subseteq} f_{pu}^{-1}(H)$. Hence, $f_{pu}(F) = f_{pu}(f_{pu}^{-1}(H)) \widetilde{\subseteq} H$. It follows that $f_{pu}(F) = H$. Hence, $f_{pu}(F) \in \min(Y, \sigma, B)$. \square

Corollary 4.5. Let $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ be bijective and soft open. If $G \in \min(Y, \sigma, B)$ with $f_{pu}^{-1}(G) \in \tau$, then $f_{pu}^{-1}(G) \in \min(X, \tau, A)$.

Theorem 4.6. Let (X, τ, A) be a STS and let B be a soft base for τ . Then $\min(X, \tau, A) \subseteq B$.

Proof. Let $F \in \min(X, \tau, A)$. Choose $a_x \widetilde{\in} F$. Since B is a soft base for τ , then by Proposition 2.29, there exists $G \in B$ such that $a_x \widetilde{\in} G \widetilde{\subseteq} F$. Since $F \in \min(X, \tau, A)$ and $G \neq 0_A$, then $F = G$. Hence, $F \in B$. \square

Theorem 4.7. If $F \in \min(X, \tau, A)$, then for every $a \in A$ with $F(a) \neq \emptyset$, $F(a) \in \min(X, \tau_a)$.

Proof. Suppose $F \in \min(X, \tau, A)$ and let $a \in A$ such that $F(a) \neq \emptyset$. Suppose $M \in \tau_a - \{\emptyset\}$ with $M \subseteq F(a)$. Take $G \in \tau$ such that $M = G(a)$. Choose $x \in G(a)$. Then $a_x \widetilde{\in} G \widetilde{\cap} F$. Since $F \in \min(X, \tau, A)$, then by Proposition 4.2 we have $F \widetilde{\subseteq} G$. Therefore, $F(a) \subseteq G(a) = M$. Hence, $M = F(a)$. This shows that $F(a) \in \min(X, \tau_a)$. \square

Lemma 4.8. [16] Let (X, τ, A) be a STS and let B be a soft base for τ . Then for every $a \in A$, the family $\{F(a) : F \in B\}$ forms a base for the topology τ_a on X .

The following example shows that the converse of Theorem 4.7 is not true in general:

Example 4.9. Let $X = \{x_1, x_2, x_3, x_4\}$ and $A = \{a_1, a_2\}$. Let

- $F_1 = \{(a_1, \{x_1\}), (a_2, \{x_1, x_2\})\}$,
- $F_2 = \{(a_1, \{x_2\}), (a_2, \{x_1, x_2\})\}$,
- $F_3 = \{(a_1, \{x_3\}), (a_2, \{x_3, x_4\})\}$,
- $F_4 = \{(a_1, \{x_4\}), (a_2, \{x_3, x_4\})\}$,
- $F_5 = \{(a_1, \emptyset), (a_2, \{x_1, x_2\})\}$,
- $F_6 = \{(a_1, \emptyset), (a_2, \{x_3, x_4\})\}$.

Let $B = \{F_1, F_2, F_3, F_4, F_5, F_6\}$ and let τ be the soft topology on X relative to A having B as a soft base. For every $i = 1, 2$, set $B_{a_i} = \{F(a_i) : F \in B\}$. By Lemma 4.8, $B_{a_1} = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\}$ is a base for τ_{a_1} and $B_{a_2} = \{\{x_1, x_2\}, \{x_3, x_4\}\}$ is a base for τ_{a_2} . Now $F_1(a_1) \in \min(X, \tau_{a_1})$ and $F_1(a_2) \in \min(X, \tau_{a_2})$, however, $F_1 \notin \min(X, \tau, A)$ because $F_5 \in \tau - \{0_A\}$ with $F_5 \subseteq F_1$ but $F_5 \neq F_1$.

Definition 4.10. A collection $B \subseteq SS(X, A)$ is called a soft partition of 1_A if the following three conditions hold:

- (a) $B \subseteq SS(X, A) - \{0_A\}$.
- (b) If $F, G \in B$, then $F \cap G = 0_A$ or $F = G$.
- (c) $\tilde{\cup}\{F : F \in B\} = 1_A$.

Theorem 4.11. For any STS (X, τ, A) , the following are equivalent:

- (a) $\tilde{\cup}\{F : F \in \min(X, \tau, A)\} = 1_A$.
- (b) $\min(X, \tau, A)$ is a soft partition of 1_A .
- (c) $\min(X, \tau, A)$ is a soft base of (X, τ, A) .

Proof. (a) \implies (b): By the definition of minimal soft open sets, $\min(X, \tau, A) \subseteq SS(X, A) - \{0_A\}$. Also, by Proposition 4.3 we have $F \cap G = 0_A$ or $F = G$ for all $F, G \in \min(X, \tau, A)$. The assumption that $\tilde{\cup}\{F : F \in \min(X, \tau, A)\} = 1_A$ ends the proof that $\min(X, \tau, A)$ is a soft partition of 1_A .

(b) \implies (c): We apply Proposition 2.29. By the definition of minimal soft open sets we have $\min(X, \tau, A) \subseteq \tau$. Let $G \in \tau - \{0_A\}$. Choose $a_x \in G$. Then $a_x \in 1_A$. Since, by (b), $\min(X, \tau, A)$ is a soft partition of 1_A , then $\tilde{\cup}\{F : F \in \min(X, \tau, A)\} = 1_A$, and so there is $F \in \min(X, \tau, A)$ such that $a_x \in F$. Since $a_x \in F \cap G$, then by Proposition 4.2 we have $F \subseteq G$. This shows that $\min(X, \tau, A)$ is a soft base of (X, τ, A) .

(c) \implies (a): Obvious. \square

Theorem 4.12. If $B \subseteq SS(X, A)$ is a soft partition of 1_A , then for every $a \in A$ the set $\{F(a) : F \in B \text{ and } F(a) \neq \emptyset\}$ is a soft partition of X .

Proof. Straightforward. \square

Theorem 4.13. Let X be an initial universe and let A be a set of parameters. Let $\{\mathfrak{S}_a : a \in A\}$ be an indexed family of topologies on X . Then $\min\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right) = \{a_Y : a \in A \text{ and } Y \in \min(X, \mathfrak{S}_a)\}$.

Proof. Let $F \in \min\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$. By Theorems 3.5 and 4.6, there is $a \in A$ and $Y \in \mathfrak{S}_a$ such that $F = a_Y$. Since $F \in \min\left(X, \bigoplus_{a \in A} \mathfrak{S}_a, A\right)$, then it is clear that $Y \in \min(X, \mathfrak{S}_a)$. The other inclusion is straightforward. \square

Theorem 4.14. If (X, \mathfrak{S}) is a topological space and A is a set of parameters, then $\min(X, C(\mathfrak{S}), A) = \{C_Y : Y \in \min(X, \mathfrak{S})\}$.

Proof. Let $C_Y \in \min(X, C(\mathfrak{S}), A)$ where $Y \in \mathfrak{S} - \{\emptyset\}$. Let $Z \in \mathfrak{S} - \{\emptyset\}$ with $Z \subseteq Y$. Then we have $C_Z \subseteq C_Y$ with $C_Z \in C(\mathfrak{S}) - \{0_A\}$ and so $C_Z = C_Y$. Thus, $Z = Y$ and hence $Y \in \min(X, \mathfrak{S})$. Conversely, let $Y \in \min(X, \mathfrak{S})$. Then $C_Y \in C(\mathfrak{S}) - \{0_A\}$. Let $C_Z \in C(\mathfrak{S}) - \{0_A\}$ where $Z \in \mathfrak{S}$ and $C_Z \subseteq C_Y$. So, we have $Z \subseteq Y$ with $Z \in \mathfrak{S} - \{\emptyset\}$ and hence $Z = Y$. It follows that $C_Z = C_Y$ and hence $C_Y \in \min(X, C(\mathfrak{S}), A)$. \square

Theorem 4.15. If (X, τ, A) is a STS with $\tau \subseteq CSS(X, A)$, then $\min(X, D(\tau)) = \{Y : C_Y \in \min(X, \tau, A)\}$.

Proof. Similar to the proof of Theorem 4.14. \square

5. Soft homogeneity

In this section, we introduce and investigate soft homogeneity in soft topological spaces; in particular, we study the relation between soft homogeneity of a soft topological space and some ordinary topological spaces generated by this soft topological space, we study the relation between homogeneity of an ordinary topological space and some soft topological spaces generated by this ordinary topological space, and we introduce some properties of homogeneous soft topological spaces that contains a minimal soft open set.

Definition 5.1. A STS (X, τ, A) is called soft homogeneous if for any $a_x, b_y \in SP(X, A)$, there exists a soft homeomorphism $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ such that $f_{pu}(a_x) = b_y$.

Theorem 5.2. Let (X, τ, A) be a STS. Then the following are equivalent:

- (a) (X, τ, A) is soft homogeneous.
- (b) For any $a_x, b_y \in SP(X, A)$, there is a soft homeomorphism $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ such that $p(x) = y$ and $u(a) = b$.
- (c) For any two pairs $(x, a), (y, b) \in X \times A$, there is a soft homeomorphism $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ such that $p(x) = y$ and $u(a) = b$.

Proof. Follows directly from the definition. \square

Example 5.3. Let $X = \{1, 2, 3, 4\}$, $A = \{a, b\}$, $F = \{(a, \{1, 2\}), (b, \{3, 4\})\}$, $G = \{(a, \{3, 4\}), (b, \{1, 2\})\}$ and $\tau = \{0_A, 1_A, F, G\}$. Then (X, τ, A) is soft homogeneous.

Proof. Let $c_x, d_y \in SP(X, A) = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$.

Case 1. $c_x = a_1$ and $d_y = a_2$. Take $u = \{(a, a), (b, b)\}$ and $p = \{(1, 2), (2, 1), (3, 3), (4, 4)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = G$ and $f_{pu}(G) = F$.
 $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 2. $c_x = a_1$ and $d_y = a_3$. Take $u = \{(a, a), (b, b)\}$ and $p = \{(1, 3), (2, 4), (3, 1), (4, 2)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = G$ and $f_{pu}(G) = F$.
 $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 3. $c_x = a_1$ and $d_y = b_1$. Take $u = \{(a, b), (b, a)\}$ and $p = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = G$ and $f_{pu}(G) = F$.
 $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 4. $c_x = a_1$ and $d_y = b_2$. Take $u = \{(a, b), (b, a)\}$ and $p = \{(1, 2), (2, 1), (3, 3), (4, 4)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = G$ and $f_{pu}(G) = F$.
 $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 5. $c_x = a_1$ and $d_y = b_3$. Take $u = \{(a, b), (b, a)\}$ and $p = \{(1, 3), (2, 4), (3, 1), (4, 2)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = G$ and $f_{pu}(G) = F$.
 $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 6. $c_x = a_1$ and $d_y = b_4$. Take $u = \{(a, b), (b, a)\}$ and $p = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = F$ and $f_{pu}(G) = G$.
 $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$. \square

Each of the rest cases is similar to one of the above six cases.

Example 5.4. Let $X = \{1, 2, 3, 4\}$ and $A = \{a, b\}$. Let

$$F = \{(a, \{1, 2\}), (b, \emptyset)\},$$

$$G = \{(a, \{3, 4\}), (b, \emptyset)\},$$

$$H = \{(a, \emptyset), (b, \{1, 2\})\},$$

$$K = \{(a, \emptyset), (b, \{3, 4\})\},$$

and $B = \{F, G, H, K\}$. The soft topological space (X, τ, A) which has B as a soft base is soft homogeneous.

Proof. Let $c_x, d_y \in SP(X, A) = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$.

Case 1. $c_x = a_1$ and $d_y = a_2$. Take $u = \{(a, a), (b, b)\}$ and $p = \{(1, 2), (2, 1), (3, 3), (4, 4)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = F$, $f_{pu}(G) = G$, $f_{pu}(H) = H$ and $f_{pu}(K) = K$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 2. $c_x = a_1$ and $d_y = a_3$. Take $u = \{(a, a), (b, b)\}$ and $p = \{(1, 3), (2, 4), (3, 1), (4, 2)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = G$, $f_{pu}(G) = F$, $f_{pu}(H) = K$ and $f_{pu}(K) = H$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 3. $c_x = a_1$ and $d_y = b_1$. Take $u = \{(a, b), (b, a)\}$ and $p = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = H$, $f_{pu}(G) = K$, $f_{pu}(H) = F$ and $f_{pu}(K) = G$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 4. $c_x = a_1$ and $d_y = b_2$. Take $u = \{(a, b), (b, a)\}$ and $p = \{(1, 2), (2, 1), (3, 3), (4, 4)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = H$, $f_{pu}(G) = K$, $f_{pu}(H) = F$ and $f_{pu}(K) = G$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 5. $c_x = a_1$ and $d_y = b_3$. Take $u = \{(a, b), (b, a)\}$ and $p = \{(1, 3), (2, 4), (3, 1), (4, 2)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = K$, $f_{pu}(G) = H$, $f_{pu}(H) = G$ and $f_{pu}(K) = F$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$. \square

Each of the rest cases is similar to one of the above five cases.

Example 5.5. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $A = \{a, b, c\}$. Let

$$F = \{(a, \{1, 2\}), (b, \{3, 4\}), (c, \{5, 6\})\},$$

$$G = \{(a, \{3, 4\}), (b, \{5, 6\}), (c, \{1, 2\})\},$$

$$H = \{(a, \{5, 6\}), (b, \{1, 2\}), (c, \{3, 4\})\},$$

and $B = \{F, G, H\}$. The soft topological space (X, τ, A) which has B as a soft base is soft homogeneous.

Proof. Let $c_x, d_y \in SP(X, A) = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6\}$.

Case 1. $c_x = a_1$ and $d_y = a_2$. Take $u = \{(a, a), (b, b), (c, c)\}$ and $p = \{(1, 2), (2, 1), (3, 3), (4, 4), (5, 5), (6, 6)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = F$, $f_{pu}(G) = G$, $f_{pu}(H) = H$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 2. $c_x = a_1$ and $d_y = a_3$. Take $u = \{(a, a), (b, b), (c, c)\}$ and $p = \{(1, 3), (2, 4), (3, 5), (4, 6), (5, 1), (6, 2)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = G$, $f_{pu}(G) = H$, $f_{pu}(H) = F$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1}(F) = H$, $f_{pu}^{-1}(G) = F$, $f_{pu}^{-1}(H) = G$.

Case 3. $c_x = a_1$ and $d_y = b_1$. Take $u = \{(a, b), (b, a), (c, c)\}$ and $p = \{(1, 1), (2, 2), (3, 5), (4, 6), (5, 3), (6, 4)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = H$, $f_{pu}(G) = G$, $f_{pu}(H) = F$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 4. $c_x = a_1$ and $d_y = b_2$. Take $u = \{(a, b), (b, a), (c, c)\}$ and $p = \{(1, 2), (2, 1), (3, 5), (4, 6), (5, 3), (6, 4)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = H$, $f_{pu}(G) = G$, $f_{pu}(H) = F$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 5. $c_x = a_1$ and $d_y = b_3$. Take $u = \{(a, b), (b, a), (c, c)\}$ and $p = \{(1, 3), (2, 4), (3, 1), (4, 2), (5, 5), (6, 6)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = F$, $f_{pu}(G) = H$, $f_{pu}(H) = F$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$.

Case 6. $c_x = a_1$ and $d_y = b_5$. Take $u = \{(a, b), (b, a), (c, c)\}$ and $p = \{(1, 5), (2, 6), (3, 3), (4, 4), (5, 1), (6, 2)\}$. Then f_{pu} is a bijection with $f_{pu}(c_x) = d_y$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft open: $f_{pu}(F) = G$, $f_{pu}(G) = F$, $f_{pu}(H) = G$.

$f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ is soft continuous: $f_{pu}^{-1} = f_{p^{-1}u^{-1}} = f_{pu}$. \square

Each of the rest cases is similar to one of the above six cases.

Theorem 5.6. For any non-empty set X and for any set of parameters A , the STS $(X, SS(X, A), A)$ is soft homogeneous.

Proof. Let $a_x, b_y \in SP(X, A)$. Define $p : X \rightarrow X$ and $u : A \rightarrow A$ as follows:

$$p(t) = \begin{cases} y & \text{if } t = x \\ x & \text{if } t = y \\ t & \text{if } t \neq x \text{ and if } t \neq y \end{cases}$$

and

$$u(e) = \begin{cases} b & \text{if } e = a \\ a & \text{if } e = b \\ e & \text{if } e \neq a \text{ and if } e \neq b \end{cases}.$$

Then f_{pu} is a soft homeomorphism with $f_{pu}(a_x) = b_y$. \square

Theorem 5.7. Let (X, τ, A) be a homogeneous STS. If there is $a_x \in SP(X, A) \cap \tau$, then $\tau = SS(X, A)$.

Proof. It is sufficient to see that $SP(X, A) \subseteq \tau$. Let $b_y \in SP(X, A)$. Since (X, τ, A) is soft homogeneous, then there is a soft homeomorphism $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ such that $f_{pu}(a_x) = b_y$. Since $a_x \in \tau$, then $b_y \in \tau$. It follows that $SP(X, A) \subseteq \tau$. \square

Corollary 5.8. Let (X, τ, A) be a STS such that there is $a_x \in SP(X, A) \cap \tau$. Then (X, τ, A) is soft homogeneous iff $\tau = SS(X, A)$.

Proof. Follows directly from Theorems 5.6 and 5.7. \square

Definition 5.9. Let X be a non-empty set and let α be a cardinal number. A partition \mathcal{M} of X is called an α -partition of X if $\alpha = |M|$ for all $M \in \mathcal{M}$.

Definition 5.10. Let (X, \mathfrak{S}) be a topological space, \mathcal{M} be a base of (X, \mathfrak{S}) and let α be a cardinal number. Then \mathcal{M} is called an α -partition base of (X, \mathfrak{S}) if \mathcal{M} is an α -partition of X .

Theorem 5.11. [33] Let (X, \mathfrak{S}) be a topological space which contains $N \in \min(X, \mathfrak{S})$. Then (X, \mathfrak{S}) is homogeneous iff (X, \mathfrak{S}) has an $|N|$ -partition base.

Theorem 5.12. [18] Let $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ be soft continuous. Then for every $a \in A$, $p : (X, \tau_a) \rightarrow (Y, \sigma_{u(a)})$ is continuous.

Corollary 5.13. Let $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ be a soft homeomorphism. Then for every $a \in A$, $p : (X, \tau_a) \rightarrow (Y, \sigma_{u(a)})$ is a homeomorphism.

Theorem 5.14. If (X, τ, A) is soft homogeneous, then for all $a \in A$, (X, τ_a) is homogeneous.

Proof. Let $a \in A$. Let $x, y \in X$. Since (X, τ, A) is soft homogeneous, then there is a soft homeomorphism $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ such that $f_{pu}(a_x) = a_y$. Then $p(x) = y$ and $u(a) = a$. By Corollary 5.13, $p : (X, \tau_a) \rightarrow (Y, \tau_a)$ is a homeomorphism with $p(x) = y$. It follows that (X, τ_a) is homogeneous. \square

Theorem 5.15. If (X, τ, A) is soft homogeneous, then for all $a, b \in A$, (X, τ_a) and (X, τ_b) are homeomorphic.

Proof. Let $a, b \in A$. Choose $x \in X$. Since (X, τ, A) is soft homogeneous, then there is a soft homeomorphism $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ such that $f_{pu}(a_x) = b_x$. Then $u(a) = b$. By Corollary 5.13, $p : (X, \tau_a) \rightarrow (X, \tau_b)$ is a homeomorphism. It follows that (X, τ_a) and (X, τ_b) are homeomorphic. \square

Each of the following two examples show that the converse of Theorem 5.14 is not true in general:

Example 5.16. Let $X = \{1, 2, 3, 4\}$ and $A = \{a, b\}$. Let

- $F = \{(a, \{1\}), (b, \{1, 2\})\}$,
- $G = \{(a, \{2\}), (b, \{1, 2\})\}$,
- $H = \{(a, \{3\}), (b, \{3, 4\})\}$,
- $K = \{(a, \{4\}), (b, \{3, 4\})\}$,
- $L = \{(a, \emptyset), (b, \{1, 2\})\}$,
- $M = \{(a, \emptyset), (b, \{3, 4\})\}$,

$B = \{1_A, F, G, H, K, L, M\}$, and let (X, τ, A) be the STS that has B as a soft base.

1. τ_a is the discrete topology and $\tau_b = \{\emptyset, X, \{1, 2\}, \{3, 4\}\}$, and by Theorem 5.11 both of (X, τ_a) and (X, τ_b) are homogeneous.

2. Since it is clear that (X, τ_a) and (X, τ_b) are not homeomorphic, then by Theorem 5.15, (X, τ, A) is not soft homogeneous.

Example 5.17. Let $X = \mathbb{R}$ and $A = \{a, b\}$. Let \mathfrak{S}_1 and \mathfrak{S}_2 be the usual and the discrete topologies on \mathbb{R} , respectively. Let $\tau = \{F \in SS(X, A) : F(a) \in \mathfrak{S}_1 \text{ and } F(b) \in \mathfrak{S}_2\}$.

1. $\tau_a = \mathfrak{S}_1$ and $\tau_b = \mathfrak{S}_2$. It is well known that both of (X, τ_a) and (X, τ_b) are homogeneous.

2. Since (X, τ_a) and (X, τ_b) are not homeomorphic, then by Theorem 5.15, (X, τ, A) is not soft homogeneous.

Lemma 5.18. Let $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ be a soft mapping. Then for every $C_Z \in CSS(Y, B)$, $f_{pu}^{-1}(C_Z) = C_{p^{-1}(Z)}$.

Proof. Let $C_Z \in CSS(Y, B)$ and let $a \in A$. Then

$$\begin{aligned} (f_{pu}^{-1}(C_Z))(a) &= p^{-1}(C_Z(u(a))) \\ &= p^{-1}(Z). \end{aligned}$$

Therefore, $f_{pu}^{-1}(C_Z) = C_{p^{-1}(Z)}$. \square

Theorem 5.19. Let $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ be a soft mapping with $\tau \subseteq CSS(X, A)$ and $\sigma \subseteq CSS(Y, B)$. Then f_{pu} is soft continuous iff $p : (X, D(\tau)) \rightarrow (Y, D(\sigma))$ is continuous.

Proof. Necessity. Suppose that f_{pu} is soft continuous. Let $Z \in D(\sigma)$. Then $C_Z \in \sigma$ and so $f_{pu}^{-1}(C_Z) \in \tau$. By Lemma 5.18, $f_{pu}^{-1}(C_Z) = C_{p^{-1}(Z)}$. Thus, $p^{-1}(Z) \in D(\tau)$. Hence, $p : (X, D(\tau)) \rightarrow (Y, D(\sigma))$ is continuous.

Sufficiency. Suppose that $p : (X, D(\tau)) \rightarrow (Y, D(\sigma))$ is continuous. Let $C_Z \in \sigma$. Then $Z \in D(\sigma)$ and so $p^{-1}(Z) \in D(\tau)$. Thus, we have $C_{p^{-1}(Z)} \in \tau$. By Lemma 5.18, $C_{p^{-1}(Z)} = f_{pu}^{-1}(C_Z)$. Thus, $f_{pu}^{-1}(C_Z) \in \tau$. Therefore, f_{pu} is soft continuous. \square

Corollary 5.20. Let $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ be a soft mapping with $\tau \subseteq CSS(X, A)$ and $\sigma \subseteq CSS(Y, B)$. Let $u : A \rightarrow B$ be a bijection. Then f_{pu} is a soft homeomorphism iff $p : (X, D(\tau)) \rightarrow (Y, D(\sigma))$ is a homeomorphism.

Theorem 5.21. Let (X, τ, A) be a STS with $\tau \subseteq CSS(X, A)$. Then (X, τ, A) is soft homogeneous iff $(X, D(\tau))$ is homogeneous.

Proof. Necessity. Suppose that (X, τ, A) is soft homogeneous. Let $x, y \in X$. Choose $a \in A$. Then there is a soft homeomorphism $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ such that $f_{pu}(a_x) = a_y$. So, $p(x) = y$. Also by Corollary 5.20, $p : (X, D(\tau)) \rightarrow (Y, D(\tau))$ is a homeomorphism with $p(x) = y$. Therefore, $(X, D(\tau))$ is homogeneous.

Sufficiency. Suppose that $(X, D(\tau))$ is homogeneous. Let $a_x, b_y \in SP(X, A)$. Choose a homeomorphism $p : (X, D(\tau)) \rightarrow (X, D(\tau))$ such that $p(x) = y$. Choose a bijection $u : A \rightarrow A$ such that $u(a) = b$. By Corollary 5.20, $f_{pu} : (X, D(\tau), A) \rightarrow (X, D(\tau), A)$ is a soft homeomorphism. Therefore, by Theorem 5.2, it follows that $(X, D(\tau), A)$ is soft homogeneous. \square

Theorem 5.22. Let $P : (X, \mathfrak{S}) \rightarrow (Y, \mathfrak{R})$ be a function. Let $u : A \rightarrow B$ be a function between two sets of parameters A and B . Then $f_{pu} : (X, C(\mathfrak{S}), A) \rightarrow (Y, C(\mathfrak{R}), B)$ is soft continuous iff p is continuous.

Proof. Necessity. Suppose that $f_{pu} : (X, C(\mathfrak{S}), A) \rightarrow (Y, C(\mathfrak{R}), B)$ is soft continuous. Choose $a \in A$. Then by Theorem 5.12, $p : (X, (C(\mathfrak{S}))_a) \rightarrow (Y, (C(\mathfrak{R}))_{u(a)})$ is continuous. By Theorem 3.17, $(C(\mathfrak{S}))_a = \mathfrak{S}$ and $(C(\mathfrak{R}))_{u(a)} = \mathfrak{R}$. It follows that $P : (X, \mathfrak{S}) \rightarrow (Y, \mathfrak{R})$ is continuous.

Sufficiency. Suppose that $P : (X, \mathfrak{S}) \rightarrow (Y, \mathfrak{R})$ is continuous. Let $C_Z \in C(\mathfrak{R})$. Then $Z \in \mathfrak{R}$ and so $p^{-1}(Z) \in \mathfrak{S}$. Thus, we have $C_{p^{-1}(Z)} \in C(\mathfrak{S})$. By Lemma 5.18, $C_{p^{-1}(Z)} = f_{pu}^{-1}(C_Z)$. Thus, $f_{pu}^{-1}(C_Z) \in \tau$. Therefore, f_{pu} is soft continuous. \square

Corollary 5.23. Let $P : (X, \mathfrak{S}) \rightarrow (Y, \mathfrak{R})$ be a function. Let $u : A \rightarrow B$ be a bijection between two sets of parameters A and B . Then $f_{pu} : (X, C(\mathfrak{S}), A) \rightarrow (Y, C(\mathfrak{R}), B)$ is a soft homeomorphism iff p is a homeomorphism.

Theorem 5.24. Let (X, \mathfrak{S}) be a topological space and let A be a set of parameters. Then $(X, C(\mathfrak{S}), A)$ is soft homogeneous iff (X, \mathfrak{S}) is homogeneous.

Proof. Necessity. Suppose that $(X, C(\mathfrak{F}), A)$ is soft homogeneous. Let $x, y \in X$. Choose $a \in A$. Then there is a soft homeomorphism $f_{pu} : (X, C(\mathfrak{F}), A) \rightarrow (X, C(\mathfrak{F}), A)$ such that $f_{pu}(a_x) = a_y$. So, $p(x) = y$. Also by Corollary 5.23, $p : (X, \mathfrak{F}) \rightarrow (X, \mathfrak{F})$ is a homeomorphism with $p(x) = y$. Therefore, (X, \mathfrak{F}) is homogeneous.

Sufficiency. Suppose that (X, \mathfrak{F}) is homogeneous. Let $a_x, b_y \in SP(X, A)$. Choose a homeomorphism $p : (X, \mathfrak{F}) \rightarrow (X, \mathfrak{F})$ such that $p(x) = y$. Choose a bijection $u : A \rightarrow A$ such that $u(a) = b$. By Corollary 5.23, $f_{pu} : (X, C(\mathfrak{F}), A) \rightarrow (X, C(\mathfrak{F}), A)$ is a soft homeomorphism. Therefore, by Theorem 5.2, it follows that $(X, C(\mathfrak{F}), A)$ is soft homogeneous. \square

Example 5.25. Consider the topological space $(\mathbb{R}^n, \mathfrak{F})$ where \mathfrak{F} is the usual topology on \mathbb{R}^n and let $A = \mathbb{R}$. It is well known that $(\mathbb{R}^n, \mathfrak{F})$ is homogeneous. Then by Theorem 5.24, $(\mathbb{R}^n, C(\mathfrak{F}), A)$ is soft homogeneous.

Example 5.26. Consider the topological space $([0, 1], \mathfrak{F})$ where \mathfrak{F} is the usual topology on $[0, 1]$ and let $A = \mathbb{N}$. It is well known that $([0, 1], \mathfrak{F})$ is not homogeneous. Then by Theorem 5.24, $([0, 1], C(\mathfrak{F}), A)$ is not soft homogeneous.

Example 5.27. Consider the topological space (S^1, \mathfrak{F}) where $S^1 = \{(x, y) : x, y \in \mathbb{R} \text{ and } x^2 + y^2 = 1\}$ and \mathfrak{F} is the usual topology on S^1 , and let $A = \mathbb{Z}$. It is well known that (S^1, \mathfrak{F}) is homogeneous. Then by Theorem 5.24, $(S^1, C(\mathfrak{F}), A)$ is soft homogeneous.

Example 5.28. Consider the topological space (X, \mathfrak{F}) where $X = \{(x, y) : x, y \in \mathbb{R} \text{ and } x^2 + y^2 \leq 1\}$ and \mathfrak{F} is the usual topology on X , and let $A = \mathbb{Z}$. It is well known that (X, \mathfrak{F}) is not homogeneous. Then by Theorem 5.24, $(X, C(\mathfrak{F}), A)$ is not soft homogeneous.

Theorem 5.29. Let X be an initial universe and let A be a set of parameters. Let $\{\mathfrak{F}_a : a \in A\}$ be an indexed family of topologies on X . If

- (a) $(X, \bigoplus_{a \in A} \mathfrak{F}_a)$ is soft homogeneous, then
- (b) (X, \mathfrak{F}_b) is homogeneous for all $b \in A$.
- (c) (X, \mathfrak{F}_b) is homeomorphic to (X, \mathfrak{F}_c) for all $b, c \in A$.

Proof. (a) Let $b \in A$. "Then by Theorem 5.14, $(X, (\bigoplus_{a \in A} \mathfrak{F}_a)_b)$ is homogeneous. By Theorem 3.7 we have $(\bigoplus_{a \in A} \mathfrak{F}_a)_b = \mathfrak{F}_b$ and hence (X, \mathfrak{F}_b) is homogeneous.

(b) Let $a, b \in A$. Then by Theorem 5.15, $(X, (\bigoplus_{a \in A} \mathfrak{F}_a)_b)$ and $(X, (\bigoplus_{a \in A} \mathfrak{F}_a)_c)$ are homeomorphic. By Theorem 3.7 we have $(\bigoplus_{a \in A} \mathfrak{F}_a)_b = \mathfrak{F}_b$ and $(\bigoplus_{a \in A} \mathfrak{F}_a)_c = \mathfrak{F}_c$. It follows that (X, \mathfrak{F}_b) is homeomorphic to (X, \mathfrak{F}_c) . \square

Question 5.30. Let $\{(X, \mathfrak{F}_a) : a \in A\}$ be an indexed family of topological spaces such that

- (a) (X, \mathfrak{F}_b) is homogeneous for all $b \in A$.
 - (b) (X, \mathfrak{F}_b) is homeomorphic to (X, \mathfrak{F}_c) for all $b, c \in A$.
- Is it true that $(X, \bigoplus_{a \in A} \mathfrak{F}_a)$ is soft homogeneous?

Theorem 5.31. Let $p : (X, \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ be a function between two topological spaces and let $u : A \rightarrow B$ be a function. Then $f_{pu} : (X, \tau(\mathfrak{F}_1), A) \rightarrow (Y, \tau(\mathfrak{F}_2), B)$ is soft continuous iff $p : (X, \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ is continuous.

Proof. Necessity. Suppose that $f_{pu} : (X, \tau(\mathfrak{F}_1), A) \rightarrow (Y, \tau(\mathfrak{F}_2), B)$ is soft continuous. Choose $a \in A$. Then by Theorem 5.12, $p : (X,$

$(\tau(\mathfrak{F}_1))_a) \rightarrow (Y, (\tau(\mathfrak{F}_2))_{u(a)})$ is continuous. By Corollary 3.8, $(\tau(\mathfrak{F}_1))_a = \mathfrak{F}_1$ and $(\tau(\mathfrak{F}_2))_{u(a)} = \mathfrak{F}_2$. Thus, $p : (X, \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ is continuous.

Sufficiency. Suppose that $p : (X, \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ is continuous. Let $G \in \tau(\mathfrak{F}_2)$ and let $a \in A$. Then $(f_{pu}^{-1}(G))(a) = p^{-1}(G(u(a)))$. Since $u(a) \in B$, then $G(u(a)) \in \mathfrak{F}_2$. Since $p : (X, \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ is continuous, then $p^{-1}(G(u(a))) \in \mathfrak{F}_1$ and so $(f_{pu}^{-1}(G))(a) \in \mathfrak{F}_1$. Therefore, $(f_{pu}^{-1}(G)) \in \tau(\mathfrak{F}_1)$ and hence $f_{pu} : (X, \tau(\mathfrak{F}_1), A) \rightarrow (Y, \tau(\mathfrak{F}_2), B)$ is soft continuous. \square

Theorem 5.32. Let $p : (X, \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ be a function between two topological spaces and let $u : A \rightarrow B$ be a bijection. Then $f_{pu} : (X, \tau(\mathfrak{F}_1), A) \rightarrow (Y, \tau(\mathfrak{F}_2), B)$ is a soft homeomorphism iff $p : (X, \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ is a homeomorphism.

Proof. Necessity. Suppose that $f_{pu} : (X, \tau(\mathfrak{F}_1), A) \rightarrow (Y, \tau(\mathfrak{F}_2), B)$ is a soft homeomorphism. Then $p : X \rightarrow Y$ is a bijection. By Theorem 5.31, $p : (X, \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ and $p^{-1} : (Y, \mathfrak{F}_2) \rightarrow (X, \mathfrak{F}_1)$ are continuous. It follows that $p : (X, \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ is a homeomorphism.

Sufficiency. Suppose that $p : (X, \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ is a homeomorphism. Then $p : X \rightarrow Y$ is a bijection. Since we have p and u are bijections, then $f_{pu} : (X, \tau(\mathfrak{F}_1), A) \rightarrow (Y, \tau(\mathfrak{F}_2), B)$ is a bijection. Since $p : (X, \mathfrak{F}_1) \rightarrow (Y, \mathfrak{F}_2)$ is continuous, then by Theorem 5.31, $f_{pu} : (X, \tau(\mathfrak{F}_1), A) \rightarrow (Y, \tau(\mathfrak{F}_2), B)$ is soft continuous. Since $p^{-1} : (Y, \mathfrak{F}_2) \rightarrow (X, \mathfrak{F}_1)$ is continuous, then again by Theorem 5.31, $f_{p^{-1}u^{-1}} : (Y, \tau(\mathfrak{F}_2), B) \rightarrow (X, \tau(\mathfrak{F}_1), A)$ is soft continuous. It follows that $f_{pu} : (X, \tau(\mathfrak{F}_1), A) \rightarrow (Y, \tau(\mathfrak{F}_2), B)$ is a soft homeomorphism. \square

The following result answers Question 5.30 partially:

Theorem 5.33. Let (X, \mathfrak{F}) be a topological space and let A be a set of parameters. Then $(X, \tau(\mathfrak{F}), A)$ is soft homogeneous iff (X, \mathfrak{F}) is homogeneous.

Proof. Necessity. Suppose that $(X, \tau(\mathfrak{F}), A)$ is soft homogeneous. For all $a \in A$, put $\mathfrak{F}_a = \mathfrak{F}$. Then $\tau(\mathfrak{F}) = \bigoplus_{a \in A} \mathfrak{F}_a$ and so $(X, \bigoplus_{a \in A} \mathfrak{F}_a)$ is soft homogeneous. By Theorem 5.29 (a), it follows that (X, \mathfrak{F}) is homogeneous.

Sufficiency. Suppose that (X, \mathfrak{F}) is homogeneous. Let $a_x, b_y \in SP(X, A)$. Choose a homeomorphism $p : (X, \mathfrak{F}) \rightarrow (X, \mathfrak{F})$ such that $p(x) = y$. Choose a bijection $u : A \rightarrow A$ such that $u(a) = b$. By Theorem 5.32, $f_{pu} : (X, \tau(\mathfrak{F}), A) \rightarrow (X, \tau(\mathfrak{F}), A)$ is a soft homeomorphism. Therefore, by Theorem 5.2, it follows that $(X, \tau(\mathfrak{F}), A)$ is soft homogeneous. \square

Example 5.34. Consider the topological space $(\mathbb{R}^n, \mathfrak{F})$ where \mathfrak{F} is the usual topology on \mathbb{R}^n and let $A = \mathbb{R}$. It is well known that $(\mathbb{R}^n, \mathfrak{F})$ is homogeneous. Then by Theorem 5.33, $(\mathbb{R}^n, \tau(\mathfrak{F}), A)$ is soft homogeneous.

Example 5.35. Consider the topological space (X, \mathfrak{F}) where X is the Cantor set and \mathfrak{F} is the usual topology on X , and let $A = \mathbb{Z}$. It is well known that (X, \mathfrak{F}) is homogeneous. Then by Theorem 5.33, $(X, \tau(\mathfrak{F}), A)$ is soft homogeneous.

For every $F \in SS(X, A)$, the set $Supp(F) = \{a \in A : F(a) \neq \emptyset\}$ is called the support of F .

Theorem 5.36. Let $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ be a soft mapping and let $F \in SS(X, A)$. Then $Supp(f_{pu}(F)) = u(Supp(F))$.

Proof. To see that $Supp(f_{pu}(F)) \subseteq u(Supp(F))$, let $b \in Supp(f_{pu}(F))$. Then $(f_{pu}(F))(b) \neq \emptyset$. Note that $(f_{pu}(F))(b) = \cup \{p(F(a)) : a \in u^{-1}(b)\}$.

So, there is $a \in u^{-1}(b)$ such that $p(F(a)) \neq \emptyset$. Thus, we have $a \in \text{Supp}(F)$ and $u(a) = b$. Therefore, $b \in u(\text{Supp}(F))$. This ends the proof that $\text{Supp}(f_{pu}(F)) \subseteq u(\text{Supp}(F))$. To see that $u(\text{Supp}(F)) \subseteq \text{Supp}(f_{pu}(F))$, let $b \in u(\text{Supp}(F))$. Then there is $a \in \text{Supp}(F)$ such that $b = u(a)$. Thus, we have $p(F(a)) \neq \emptyset$ and $p(F(a)) \subseteq \cup \{p(F(a)) : a \in u^{-1}(b)\} = (f_{pu}(F))(b)$. It follows that $b \in \text{Supp}(f_{pu}(F))$. This ends the proof that $u(\text{Supp}(F)) \subseteq \text{Supp}(f_{pu}(F))$. \square

Theorem 5.37. Let $f_{pu} : (X, \tau, A) \rightarrow (Y, \sigma, B)$ be a soft mapping with $u : A \rightarrow B$ is an injection, and let $F \in \text{SS}(X, A)$. Then $|\text{Supp}(f_{pu}(F))| = |\text{Supp}(F)|$.

Proof. By Theorem 5.36, $\text{Supp}(f_{pu}(F)) = u(\text{Supp}(F))$ and so $|\text{Supp}(f_{pu}(F))| = |u(\text{Supp}(F))|$. Since $u : A \rightarrow B$ is an injection, then $|u(\text{Supp}(F))| = |\text{Supp}(F)|$. It follows that $|\text{Supp}(f_{pu}(F))| = |\text{Supp}(F)|$. \square

Theorem 5.38. Let (X, τ, A) be a homogeneous STS which contains $F \in \min(X, \tau, A)$. Then

- (a) $\min(X, \tau, A)$ is a soft base for (X, τ, A) .
 (b) For all $G \in \min(X, \tau, A)$ and for all $a, b \in A$ with $F(a) \neq \emptyset$ and $G(b) \neq \emptyset$, $|F(a)| = |G(b)|$.
 (c) For all $G \in \min(X, \tau, A)$, $|\text{Supp}(F)| = |\text{Supp}(G)|$.

Proof. (a) We apply Proposition 2.29. Clearly that $\min(X, \tau, A) \subseteq \tau$. Let $G \in \tau - \{0_A\}$ and let $b_y \in G$. Choose $a_x \in F$. Since (X, τ, A) is soft homogeneous, then there is a soft homeomorphism $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ such that $f_{pu}(a_x) = b_y$. So, $b_y \in f_{pu}(F) \cap G$. Since $F \in \min(X, \tau, A)$, then by Theorem 4.4, $f_{pu}(F) \in \min(X, \tau, A)$. Thus, by Proposition 4.2, $f_{pu}(F) \subseteq G$. Therefore, $\min(X, \tau, A)$ is a soft base for (X, τ, A) .

(b) Let $G \in \min(X, \tau, A)$ and let $a, b \in A$ such that $F(a) \neq \emptyset$ and $G(b) \neq \emptyset$. Choose $x \in F(a)$ and $y \in G(b)$. Then $a_x \in F$ and $b_y \in G$. Since (X, τ, A) is soft homogeneous, then there is a soft homeomorphism $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ such that $f_{pu}(a_x) = b_y$. Since $F \in \min(X, \tau, A)$, by Theorem 4.4, $f_{pu}(F) \in \min(X, \tau, A)$. Since $a_x \in F$, then $b_y = f_{pu}(a_x) \in f_{pu}(F)$. Since $f_{pu}(F) \in \min(X, \tau, A)$ and $b_y \in G \cap f_{pu}(F)$, then by Proposition 4.3, $f_{pu}(F) = G$. Note that $|G(b)| = |(f_{pu}(F))(b)| = |p(F(u^{-1}(b)))| = |p(F(a))|$. Since p is bijective, then $|p(F(a))| = |F(a)|$. Therefore, $|F(a)| = |G(b)|$.

(c) Choose $a_x \in F$ and $b_y \in G$. Since (X, τ, A) is soft homogeneous, then there is a soft homeomorphism $f_{pu} : (X, \tau, A) \rightarrow (X, \tau, A)$ such that $f_{pu}(a_x) = b_y$. By Theorem 4.4, $f_{pu}(F) \in \min(X, \tau, A)$. Since $b_y \in G \cap f_{pu}(F)$, then by Proposition 4.3, $f_{pu}(F) = G$. Thus, by Theorem 5.37, it follows that $|\text{Supp}(F)| = |\text{Supp}(f_{pu}(F))| = |\text{Supp}(G)|$. \square

Corollary 5.39. Let (X, τ, A) be a homogeneous STS with the property that $\text{Supp}(G) = A$ for all $G \in \tau - \{0_A\}$. If $F \in \min(X, \tau, A)$, then we have the following:

- (a) $\min(X, \tau, A)$ is a soft base for (X, τ, A) .
 (b) For all $G \in \min(X, \tau, A)$ and for all $a, b \in A$, $|F(a)| = |G(b)|$.

6. Conclusion

In this paper, homogeneity as an ordinary topological property is extended to include soft topological spaces. Also, the study of soft minimal open sets is continued. The results deals mainly with the relation between the generated soft topology and the given indexed family of topologies defined in [17]. Also, some properties of soft homogeneous soft topological spaces that contains a minimal soft open set are given. In our future study, the following topics could be considered: 1) To define soft homogeneity components; 2) To extend countable dense homogeneity to include soft topological spaces.

Declarations

Author contribution statement

Samer Al Ghour: Conceived and designed the experiments; Analyzed and interpreted the data; contributed reagents, materials, analysis tools or data; Wrote the paper.

Awatef Bin-Saadon: Performed the experiments; Analyzed and interpreted the data; contributed reagents, materials, analysis tools or data; Wrote the paper.

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