



# On mathematical structures on pairwise comparisons matrices with coefficients in an abstract group arising from quantum gravity



Jean-Pierre Magnot <sup>a,b,\*</sup>

<sup>a</sup> LAREMA, universit  d'Angers, Boulevard Lavoisier, F-49045 Angers cedex 01, France

<sup>b</sup> Lyc e Jeanne d'Arc, 30 avenue de Grande Bretagne, F-63000 Clermont-Ferrand, France

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## ABSTRACT

We describe the mathematical properties of pairwise comparisons matrices with coefficients in an arbitrary group. Inspired by the well-known mathematical structures of quantum gravity and lattice gauge theories in physics and by the application of this framework in economy by other authors, we describe how the same structures arise in pairwise comparisons. First, we provide a vocabulary adapted for the description of main algebraic properties of inconsistency maps, and give a full description of elementary properties that extend to our generalized setting. Then, we show a geometric picture where inconsistency in pairwise comparisons can be interpreted as the non-trivial Holonomy on a loop. We continue with probabilistic aspects, again adapting ideas from quantum physics where probabilities modelize uncertainty. We finish by examples where the use of a non-abelian group is necessary.

## Introduction

Pairwise comparisons are among the classical ways of decision making and information checking. The main idea is simple: assign a score to the comparison of a pair  $(s, s')$  of two states which have to be compared, and we say that the “scores” are consistent if, for three states  $(s, s', s'')$ , the comparison of  $s$  and  $s''$  can be deduced from the comparisons of  $s$  and  $s'$ , and of  $s'$  and  $s''$ . If not, the comparisons are called inconsistent. These aspects are precised in section 1.1, and gives so many applications that it is impossible to cite them all. We mention some of them [1, 2, 3] which are applications of deep interest.

This is mostly why some authors have developed ways to quantify inconsistency [4, 5], see e.g. [6], and there exists actually tentatives of axiomatizations of the so-called “inconsistency indicators” [7, 8, 9]. One can also try to deal with partial orders, after [10, 11, 12, 13], motivated by the obvious lack of informations when one tries to express a complex situation only by a, or by the necessary hierarchization of constraints in a difficult problem.

This leads us to the main motivation of this work. Dealing with partial orders can turn out to be very complex (see e.g. [13]), where as (non-abelian) groups furnish a minimal setting where multiplication and inversion are well-defined, with all the necessary properties for comparisons of more complex data. Such an approach is already used

in [14, 15, 16, 17, 18], and it has been remarked a striking analogy with so-called gauge theories in quantum physics [18, 19]. Other applied approaches can be found in [20, 21, 22, 23] which shows the great vitality of this topic for applications in various fields. However, an abstract approach of pairwise comparisons with values in a group can highlight the mathematical structures that may appear more “canonical” than others, that could be related to the very special case of the group  $\mathbb{R}_+^*$  that is used in most applications. The aim of this paper is to draw-back the structures highlighted by physics and to adapt them to the context of pairwise comparisons matrices. As a by-product, the choice of the largest possible setting is natural, since we intend here to describe new perspectives of methods, even if most of pairwise comparisons matrices have coefficients in  $\mathbb{R}_+^*$ . This is also the occasion to analyze whether classical objects in pairwise comparisons matrices arise from higher mathematical or physical considerations, or if they are only valid in the  $\mathbb{R}_+^*$ -setting. Let us now describe the contents of this paper.

Section 1.1 reviews selected topics on  $\mathbb{R}_+^*$ -pairwise comparisons matrices which give rise to natural generalizations linked with a quantum physics picture.

Section 1.2 describes pairwise comparisons matrices with coefficients in a group  $G$ , indexed by any (finite or infinite) set of states  $I$ . We define also what is an inconsistency map. The terminology of inconsistency indicator is reserved to inconsistency maps with additional

\* Correspondence to: LAREMA, universit  d'Angers, Boulevard Lavoisier, F-49045 Angers cedex 01, France.

E-mail address: [jean-pierr.magnot@ac-clermont.fr](mailto:jean-pierr.magnot@ac-clermont.fr).

properties along the lines of [24], and the settings developed will be justified by other parts of this work.

Section 1.3 deals with algebraic properties of consistency which extends as naturally as possible the classical setting  $G = \mathbb{R}_+^*$ . We highlight adjoint action of the group  $G^I$ , called gauge group (terminology justified in section 1.6). We prove that consistent PC matrices describe an orbit of this group. The side-properties are then described, which motivates the vocabulary for properties of inconsistency maps, and leads to the terminology of inconsistency indicator. Koczkodaj's inconsistency map is shown to be an adjoint-invariant inconsistency indicator.

Section 1.4 deals with generalization on graphs. This happens when two states cannot be compared by direct comparisons, but only by comparisons with intermediate states. This leads to "holes" in the pairwise comparisons matrices, assigned to the coefficient "0", and the notion of holonomy enables us to extend Koczkodaj's inconsistency maps to a  $\mathbb{R}[[X]]$ -valued inconsistency map, which is adjoint-invariant.

In section 1.5, we first answer to [25] where an approach by so-called  $G$ -distances is proposed. We feel the need to show that this approach does not seem natural for us and from the viewpoint of this work. More natural is the notion of filter, which appears here as more topological than set-theoretic since it appears in a neighbourhood-like structure of consistent PC matrices, called vicinities, inherited from inconsistency maps. We describe in details how inconsistency maps naturally give rise to a filter of so-called vicinities of the set  $CPC_I(G)$ . We precise the reasons for this terminology.

In section 1.6, motivated by Yamabe theorem that we recall briefly, we analyze some geometric aspects of pairwise comparisons. The correspondence with gauge theories then arises clearly. We consider the conditions of consistency and inconsistency in the geometric setting of a finite or infinite dimensional simplex (section 1.6.1), which can be understood as higher dimensional triangles (which are 2-simplexes) and tetrahedra (which are 3-simplexes), see e.g. [26, 27]; each  $s_i$  corresponds to a 0-vertex; each 1-vertex gives an edge, and a triad (where inconsistency can be measured) is a 2-face (or 2-simplex). According to this setting, a geometric picture which is very similar to inconsistency is the holonomy of a connection (see, for example, [28] for holonomy in finite dimensions, and [29] for the infinite dimensional case). We show that a PC matrix  $A$  can always be expressed as a holonomy matrix if the group  $G$  is exponential, and when  $G = (\mathbb{R}_+^*)^J$ , consistent PC matrices are holonomies of a flat connection which can be constructed.

We finish with probabilistic approaches and examples. Examples under consideration may be qualified as examples, highlighted to give accessible situations which mathematical intuition can be compared with. Necessity of strongest examples seem unnecessary in view of the literature given in bibliography where simplicial or lattice gauge theories (i.e. pairwise comparisons) arise. Concerning probabilistic aspects, we restrict ourselves to interpret, in terms of pairwise comparisons matrices, two probabilistic approaches of gauge theories. The first approach presented relies on cylindrical approximations of the Weiner measure, while the second approach intends to explain in a way as simple as possible a way to understand inconsistency indicators in a way parallel to Lagrangian theories and action functionals. These very technical aspects of mathematics used in physics are here simplified and adapted in the spirit of the whole paper.

## 1. Theory

### 1.1. Pairwise comparisons matrices with coefficients in $\mathbb{R}_+^*$

It is easy to explain the inconsistency in pairwise comparisons when we consider cycles of three comparisons, called triad and represented here as  $(x, y, z)$ , which do not have the "morphism of groupoid" property such as

$$x * z \neq y,$$

which reads as

$$xz \neq y$$

in the multiplicative group  $\mathbb{R}_+^*$ . Evidently, the inconsistency in a triad  $(x, y, z) \in \mathbb{R}^3$  is somehow (not linearly) proportional to  $y - xz$ . In the linear space, the inconsistency is measured by the "approximate flatness" of the triangle. The triad is consistent if the triangle is flat. For example,  $(1, 2, 1)$  and  $(10, 101, 10)$  have the difference  $y - xz = 1$  but the inconsistency in the first triad is unacceptable. It is acceptable in the second triad. In order to measure inconsistency, one usually considers coefficients  $a_{i,j}$  with values in an abelian group  $G$ , with at least 3 indexes  $i, j, k$ . The use of "inconsistency" has a meaning of a measure of inconsistency in this study; not the concept itself. The approach to inconsistency (originated in [4] and generalized in [30]) can be reduced to a simple observation:

- search all triads (which generate all 3 by 3 PC sub matrices) and locate the worst triad with a so-called inconsistency indicator ( $ii$ ),
- $ii$  of the worst triad becomes  $ii$  of the entire PC matrix.

Expressing it a bit more formally in terms of triads (the upper triangle of a PC sub matrix  $3 \times 3$ ), we have:

$$Kii(x, y, z) = 1 - \min \left\{ \frac{y}{xz}, \frac{xz}{y} \right\}. \tag{1}$$

According to [31], it is equivalent to:

$$ii(x, y, z) = 1 - e^{-\left| \ln \left( \frac{y}{xz} \right) \right|}$$

The expression  $|\ln(\frac{y}{xz})|$  is the distance of the triad  $T$  from 0. When this distance increases, the  $ii(x, y, z)$  also increases. It is important to notice here that this definition allows us to localize the inconsistency in the matrix PC and it is of a considerable importance for most applications.

Another possible definition of the inconsistency indicator can also be defined (following [31]) as:

$$Kii_n(A) = 1 - \min_{1 \leq i < j \leq n} \min \left( \frac{a_{ij}}{a_{i,i+1} a_{i+1,i+2} \dots a_{j-1,j}}, \frac{a_{i,i+1} a_{i+1,i+2} \dots a_{j-1,j}}{a_{ij}} \right) \tag{2}$$

since the matrix  $A$  is consistent if and only if for any  $1 \leq i < j \leq n$  the following equation holds:

$$a_{ij} = a_{i,i+1} a_{i+1,i+2} \dots a_{j-1,j}.$$

It is equivalent to:

$$Kii_n(A) = 1 - \max_{1 \leq i < j \leq n} \left( 1 - e^{-\left| \ln \left( \frac{a_{ij}}{a_{i,i+1} a_{i+1,i+2} \dots a_{j-1,j}} \right) \right|} \right) \tag{3}$$

The first Koczkodaj's indicator  $Kii_3$  allows us not only to find the localization of the worst inconsistency but to reduce the inconsistency by a step-by-step process which is crucial for practical applications. The second Koczkodaj's indicator  $Kii_n$  is useful when the global inconsistency indicator is needed for acceptance or rejection of the PC matrix. An abstract unification will be proposed at the end of section 1.4.

### 1.2. Changing the comparisons structure to arbitrary groups

In the previous section, the comparisons coefficients are  $a_{i,j}$  are scaling coefficients. This means that, if the PC matrix  $A$  is coherent, given a state  $s_k$ , we can recover all the other states  $s_j$  by something assimilated to scalar multiplication:

$$s_j = a_{j,k} s_k.$$

In other words, even if the states  $s_j$  are driven by more complex rules, we reduce them to a "score" or an "evaluation" in  $\mathbb{R}_+$ . This is useless to say that such an approach is highly reductive: even in video games,

the virtual fighters have more than one characteristic: health, speed, strength, mental... and the global design of these characteristics intends to reflect some “complexity” in the game (please note the “ ”). So that, the states  $s_j$  have to belong to a more complex state space  $S$ , and in order to have pairwise comparisons, a straightforward study shows that we define [32, 33] a semi-category  $C_S$ , with set of objects  $Ob(C_S) = S$ , and such that morphisms  $Mor(C_S)$  must satisfy the following properties:

- there exists an identity morphism,
- if  $a \in Hom(C_S)$ , then there exists  $a^{-1} \in Hom(C_S)$ ,
- any morphism acts on any state, which can be rephrased in the language of categories by: the semi-category is total.

Thus, gathering the necessary properties of  $Mor(C_S)$ , we get:

**Proposition 1.1.** *Hom(C<sub>S</sub>) is a group.*

This confirms the setting described in [25, section 2].

**Notations.** The last proposition shows that the group  $G$  under consideration can be any abstract group. When this is the case, in an algebraic setting, the group laws will be denoted by  $*$ . However, in sections 1.5.2, 1.6 and 1.7, we introduce additional structures, namely metric, differential geometric and probabilistic structures. In these contexts, the group laws are usually denoted with the multiplicative notation, which we will follow. In any case, the inverse of  $g \in G$  will be denoted by  $g^{-1}$ .

Let  $I$  be a set of indexes and let  $(k, +, \cdot, | \cdot |)$  be a field with absolute value and  $V_k$  a normed  $k$ -vector space.

**Definition 1.2.** [19, 34] Let  $(G, *)$  be a group. A **pairwise comparisons matrix** is a matrix

$$A = (a_{i,j})_{(i,j) \in I^2}$$

such that

- (1)  $\forall (i, j) \in I^2, a_{i,j} \in G$ .
- (2)  $\forall (i, j) \in I^2, a_{j,i} = a_{i,j}^{-1}$ .
- (3)  $a_{i,i} = e_G$  (neutral element).

We denote by  $PC_I(G)$  the set of pairwise comparisons matrices indexed by  $I$  and with coefficients in  $G$ . When  $G$  is not abelian, there are two notions of inconsistency:

- $A$  is **covariantly consistent** if and only if  $\forall (i, j, k) \in I^3, a_{i,k} = a_{i,j} * a_{j,k}$ .
- $A$  is **contravariantly consistent** if and only if  $\forall (i, j, k) \in I^3, a_{i,k} = a_{j,k} * a_{i,j}$ .

Contravariant consistency appears in the geometric realization of  $PC_I(G)$  via the holonomy of a connection on a simplex [34], but we give the following easy remark:

**Remark 1.3.** Let  $A = (a_{i,j})_{(i,j) \in I^2}$  be a contravariant PC matrix. Then the  $G$ -matrix  $B = (b_{i,j})_{(i,j) \in I^2}$  defined by

$$\forall (i, j) \in I^2, b_{i,j} = a_{i,j}^{-1}$$

is a covariant PC matrix.

This shows that the two notions are dual, and we concentrate our efforts on covariant consistency in this section, that we call **consistency**.

**Definition 1.4.** Let  $V_k$  be a vector space equipped with semi-norms. A (non-normalized, non-covariant) inconsistency map is a map

$$ii : PC_I(G) \rightarrow V_k$$

such that  $ii(A) = 0$  if  $A$  is consistent. Moreover, we say that  $ii$  is faithful if  $ii(A) = 0$  implies that  $A$  is consistent.

We denote by  $CPC_I(G)$  the set of consistent PC matrices.

After that, since  $V_k$  is a vector space equipped with a semi-norm, the semi-norm will give us the “score” of inconsistency, as in the previous section. One can assume for the sake of simplicity that  $V_k$  is a (normed) Euclidean space.

### 1.3. Algebraic properties on $PC_I(G)$

In this section only, we denote the group laws by  $*$ . First, we give the following easy proposition:

**Proposition 1.5.** *Any morphism of group  $a : G \rightarrow G'$  extends to a map  $\bar{a} : PC_I(G) \rightarrow PC_I(G')$  by action on the coefficients, and:*

- If  $A \in PC_I(G)$  is consistent, then  $\bar{a}(A) \in PC_I(G')$  is consistent.
- If  $\text{Ker}(a) = \{e_G\}$ , then  $A \in CPC_I(G)$  if and only if  $\bar{a}(A) \in CPC_I(G')$ .

We call  $G^I$  the **gauge group** of  $G$ , following [34]. Then we get the following actions:

- a left action  $L : G^I \times PC_I(G) \rightarrow PC_I(G)$  defined, for  $(g_i)_I \in G^I$  and  $(a_{i,j})_{I^2} \in PC_I(G)$  by

$$L_{(g_i)_I}((a_{i,j})_{I^2}) = (b_{i,j})_{I^2}$$

with

$$b_{i,j} = \begin{cases} 1 & \text{if } i = j \\ g_i * a_{i,j} & \text{if } i < j \\ a_{i,j} * g_j^{-1} & \text{if } i > j \end{cases}$$

- a right action  $R : PC_I(G) \times G^I \rightarrow PC_I(G)$  defined, for  $(g_i)_I \in G^I$  and  $(a_{i,j})_{I^2} \in PC_I(G)$  by

$$R_{(g_i)_I}((a_{i,j})_{I^2}) = (b_{i,j})_{I^2}$$

with

$$b_{i,j} = \begin{cases} 1 & \text{if } i = j \\ a_{i,j} * g_j & \text{if } i < j \\ g_i^{-1} * a_{i,j} & \text{if } i > j \end{cases}$$

- an adjoint action

$$Ad_{(g_i)_I} = L_{(g_i)_I} \circ R_{(g_i)_I}^{-1} = R_{(g_i)_I}^{-1} \circ L_{(g_i)_I}$$

- a coadjoint action

$$((a_{i,j})_{I^2}, (g_i)_I) \mapsto Ad_{(g_i)_I}^{-1}((a_{i,j})_{I^2}).$$

**Theorem 1.6.** [34] *When  $I \subset \mathbb{Z}$ ,*

$$\exists (\lambda_i)_{i \in I}, a_{i,j} = \lambda_i * \lambda_j^{-1} \Leftrightarrow A \text{ is consistent.}$$

We rephrase it the following way, extending it to any totally ordered set of indexes  $I$ :

**Theorem 1.7.** *Consistent PC matrices are the orbits of the PC matrix  $(e_G)_{I^2}$  with respect to the adjoint action.*

**Proof.** Let  $A = (a_{i,j})_{I^2}$  be a consistent PC matrix. Let  $i_0 \in I$  be a fixed index, and set  $g_i = a_{i,i_0}$ . Since  $A$  is consistent,

$$\begin{aligned} a_{i,j} &= a_{i,i_0} * a_{i_0,j} \\ &= a_{i,i_0} * a_{j,i_0}^{-1} \\ &= g_i * e_G * g_j^{-1} \quad \square \end{aligned}$$

Let us give the following trivial proposition:

**Proposition 1.8.**  $L$  and  $R$  are effective actions.

One can wonder whether  $L$  and  $R$  are free or transitive. Let us consider the following “layered cake” example:

Let  $\lambda \in \mathbb{R}_+^* - \{1\}$ . Let us consider the matrix

$$A = \begin{pmatrix} 1 & \lambda & \lambda^{-k} \\ \lambda^{-1} & 1 & \lambda \\ \lambda^k & \lambda^{-1} & 1 \end{pmatrix}.$$

Let us calculate the orbit of  $A$  with respect to the left action (with the special case  $G$  is abelian). Let  $g \in (\mathbb{R}_+^*)^3$ .  $L_g(A) \in CPC_3(\mathbb{R}_+^*)$ .

$$L_g(A) = \begin{pmatrix} 1 & g_1 \lambda & g_1 \lambda^{-k} \\ g_1^{-1} \lambda^{-k} & 1 & g_2 \lambda \\ g_1^{-1} \lambda^k & g_2^{-1} \lambda^{-1} & 1 \end{pmatrix}$$

In this example,  $g_3$  is not acting, so that  $L$  is not free. Let us solve

$$L_g(A) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

we get the incompatible equations:

$$\begin{cases} g_1 = \lambda^{-1} \\ g_2 = \lambda^{-1} \\ g_1 = \lambda^k \end{cases}$$

So that  $A$  is not in the orbit of  $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$  for the left action, and hence

the left action is not transitive. However, one could wonder whether the orbits of the left action intersect  $CPC_I$ . With the same example, let us try to solve “ $L_g(A)$  is consistent”, we get

$$g_1 g_2 \lambda^2 = g_1 \lambda^{-k}$$

which gives

$$g_2 = \lambda^{-2-k}.$$

This gives a one parameter family of solutions

$$(g_1, g_2) = (t, \lambda^{-2-k}), \quad t \in \mathbb{R}_+^*.$$

Generalizing this, we give:

**Theorem 1.9.** If  $I = \mathbb{N}_3$ , any orbit for the left action intersects  $CPC_3(G)$ . If  $\text{card}(I) > 3$ , there exists orbits for the left action which do not intersect  $CPC_I(G)$ .

**Proof.** Let  $A \in PC_3(G)$  and  $g = (g_1, g_2, g_3) \in G^3$

$$L_g(A) = \begin{pmatrix} 1 & g_1 * a_{1,2} & g_1 * a_{1,3} \\ a_{2,1} * g_1^{-1} & 1 & g_2 * a_{2,3} \\ a_{3,1} * g_1^{-1} & a_{3,2} * g_2^{-1} & 1 \end{pmatrix}$$

We want to find  $g$  in order to make  $L_g(A)$  consistent. We get the following relation, among others:

$$\begin{cases} g_1 * a_{1,2} * g_2 * a_{2,3} = g_1 * a_{1,3} \\ a_{2,1} * a_{1,3} = a_{2,3} * g_2^{-1} \end{cases}$$

which gives

$$g_2 = a_{2,1} * a_{1,3} * a_{3,2}$$

This condition gives the consistent PC matrix:

$$L_g(A) = \begin{pmatrix} 1 & g_1 * a_{1,2} & g_1 * a_{1,3} \\ a_{2,1} * g_1^{-1} & 1 & a_{2,1} * a_{1,3} \\ a_{3,1} * g_1^{-1} & a_{3,1} * a_{1,2} & 1 \end{pmatrix}.$$

Let us now consider  $A \in PC_4(G)$  and  $g = (g_1, g_2, g_3, g_4) \in G^4$ . Then

$$L_g(A) = \begin{pmatrix} 1 & g_1 * a_{1,2} & g_1 * a_{1,3} & g_1 * a_{1,4} \\ a_{2,1} * g_1^{-1} & 1 & g_2 * a_{2,3} & g_2 * a_{2,4} \\ a_{3,1} * g_1^{-1} & a_{3,2} * g_2^{-1} & 1 & g_3 * a_{3,4} \\ a_{4,1} * g_1^{-1} & a_{4,2} * g_2^{-1} & a_{4,2} * g_3^{-1} & 1 \end{pmatrix}.$$

We then apply the procedure given for  $3 \times 3$  PC matrices on the diagonal  $3 \times 3$  blocks. This gives, for  $i \in \{2; 3\}$ :

$$g_i = a_{i,i-1} * a_{i-1,i+1} * a_{i+1,i}$$

and reporting this equality in the matrix, we get

$$L_g(A) = \begin{pmatrix} 1 & g_1 * a_{1,2} & g_1 * a_{1,3} & g_1 * a_{1,4} \\ a_{2,1} * g_1^{-1} & 1 & a_{2,1} * a_{1,3} & a_{2,1} * a_{1,3} * a_{3,2} * a_{2,4} \\ a_{3,1} * g_1^{-1} & a_{3,1} * a_{1,2} & 1 & a_{3,2} * a_{2,4} \\ a_{4,1} * g_1^{-1} & a_{4,2} * a_{2,3} * a_{3,1} * a_{1,2} & a_{4,2} * a_{2,3} & 1 \end{pmatrix}.$$

So that, consistency now depends on the first line and the first column, and we get the relations:

$$g_1 a_{1,2} * a_{2,1} * a_{1,3} * a_{3,2} * a_{2,4} = g_1 * a_{1,4} \tag{4}$$

$$g_1 a_{1,3} * a_{3,2} * a_{2,4} = g_1 * a_{1,4} \tag{5}$$

After simplifying  $g_1$ , we gather the two lines give the same condition

$$a_{1,4} = a_{1,3} a_{3,2} a_{2,4}. \tag{6}$$

This condition is not fulfilled, unless in very special cases. For an arbitrary  $PC_I(G)$ , with  $\text{card}(I) > 4$ , we extract a  $4 \times 4$ -PC matrix to get the same result.  $\square$

Let us now turn to other properties of inconsistency maps.

**Definition 1.10.** Let  $ii$  be an inconsistency map. It is called:

- **normalized** if  $\forall A \in PC_I(G), ||ii(A)|| \leq 1$ ,
- **Ad-invariant** if  $\forall A \in PC_I(G), \forall g \in G^I, ii(Ad_g(A)) = ii(A)$ ,
- **norm invariant** if  $||ii(\cdot)||$  is Ad-invariant.

Let  $F_I(G)$  be the quotient space for the Adjoint action of the gauge group on  $PC_I(G)$ . The next result is a classical factorization theorem:

**Theorem 1.11.** An Ad-invariant inconsistency map  $ii$  factors in a unique way through the maps

$$ii = f \circ \pi$$

where

- $\pi : PC_I(G) \rightarrow F_I(G)$  is the quotient projection,
- $f \in V_k^{F_I(G)}$ .

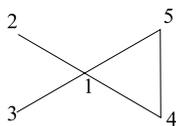


Fig. 1. The graph  $\Gamma_5$ .

We give also the following easy proposition:

**Proposition 1.12.** *Morphisms of groups are acting by pull-back on inconsistency indicators. Moreover, the pull-back of a normalized (resp. Ad-invariant) inconsistency map is a normalized (resp. Ad-invariant) inconsistency map.*

According to [9] and generalizing to any group  $G$ , we give now the following definition:

**Definition 1.13.** An inconsistency indicator  $ii$  on  $PC_I(G)$  is a faithful, normalized inconsistency map with values in  $\mathbb{R}_+$  such that there exists an inconsistency map  $ii_3$  on  $PC_3(G)$  that defines  $ii$  by the following formula

$$ii(A) = \sup \{ ii_3(B) \mid B \subset A \wedge B \in PC_3(G) \}.$$

We remark here that since  $ii$  is faithful, it is in particular (trivially) Ad-invariant on  $CPC_I(G)$ , but we do not require it to be Ad-invariant. Moreover, with such a definition, to show that  $ii$  is Ad-invariant, it is sufficient to show that  $ii_3$  is Ad-invariant. However, we give the example driven by Koczkodaj's approach. This is already proved that  $Kii_3$  generates an inconsistency indicator [9] and we complete this result by the following property:

**Proposition 1.14.** *Let  $n \geq 3$ . Koczkodaj's inconsistency maps  $Kii_3$  and  $Kii_n$  generates Ad-invariant inconsistency maps on  $PC_n(\mathbb{R}_+^*)$ .*

**Proof.** This follows from straightforward computations of the type:

$$\lambda_1 a_{1,3} \lambda_3^{-1} ((\lambda_1 a_{1,2} \lambda_2^{-1}) (\lambda_2 a_{2,3} \lambda_3^{-1}))^{-1} = a_{1,3} a_{2,3}^{-1} a_{1,2}^{-1}. \quad \square$$

1.4. Generalization: comparisons on a graph

We consider in this section a family of states  $(s_i)_I$  such that any  $s_i$  cannot be a priori compared directly with any other  $s_j$ . This leads us to consider a graph  $\Gamma_I$  linking the elements which can be compared. For example, in the previous sections,  $\Gamma_I$  was the 1-skeleton of the simplex. For simplicity, we assume that  $\Gamma_I$  is a connected graph, and that at most one vertex connects any two states  $s_i$  and  $s_j$ . We note this (oriented) vertex by  $\langle s_i, s_j \rangle$ , and the comparison coefficient by  $a_{i,j}$ . By the way, we get a pairwise comparisons matrix  $A$  indexed by  $I$  with "holes" (with virtual 0-coefficient) when a vertex does not exist, and for which  $a_{j,i}^{-1} = a_{i,j}$  when the  $(i, j)$ -vertex exists.

**Example.** Let us consider the graph  $\Gamma_5$ , with 5 states described Fig. 1. A PC matrix on  $\Gamma_5$  is of the type:

$$\begin{pmatrix} 1 & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{1,2}^{-1} & 1 & 0 & 0 & 0 \\ a_{1,3}^{-1} & 0 & 1 & 0 & 0 \\ a_{1,4}^{-1} & 0 & 0 & 1 & a_{4,5} \\ a_{1,5}^{-1} & 0 & 0 & a_{4,5}^{-1} & 1 \end{pmatrix}.$$

1.4.1. Hierarchy-less comparisons, "hearsay" evaluation and holonomy on a graph

In this model, the comparison between two states  $s_i$  and  $s_j$  can be performed by any path between  $s_i$  and  $s_j$  of any length. One can think about the propagation of rumours, where validation of information is based on hearsay results. With this approach, the capacity of propagation of an evaluation is not controlled. We denote by

$$\langle s_{i_1}, \dots, s_{i_k} \rangle = \langle s_{i_1}, s_{i_2} \rangle \vee \dots \vee \langle s_{i_{k-1}}, s_{i_k} \rangle$$

the composition of paths along vertices. By analogy with the holonomy of a connection (see section 1.6), we define:

**Definition 1.15.** Let  $s = s_i$  and  $s' = s_j$  be two states and let

$$\mathcal{H}_{s,s'} = \{ a_{i,i_2} * \dots * a_{i_{k-1},i} \mid \langle s, s_{i_2}, \dots, s_{i_{k-1}}, s' \rangle \text{ is a path from } s \text{ to } s' \}.$$

We denote by  $\mathcal{H}_s$  the set  $\mathcal{H}_{s,s}$ .

By the way, we get the following properties, usual for classical holonomy. We recall that, given a state  $s$ , then  $\mathcal{H}_s$  is a subgroup of  $G$ .

**Definition 1.16.** The holonomy group at the state  $s$  is the group  $\mathcal{H}_s$ .

**Proposition 1.17.**

- (1) Let  $s$  and  $s'$  be two states. Then  $\mathcal{H}_s$  and  $\mathcal{H}_{s'}$  are conjugate subgroups of  $G$ .
- (2)

$$\mathcal{H}_{s_i, s_j} = a_{i,j} * \mathcal{H}_{s_j} = \mathcal{H}_{s_i} * a_{i,j}.$$

**Example.** With the graph  $\Gamma_5$  of Fig. 1,  $\mathcal{H}_1$  is the subgroup of  $G$  generated by  $a_{1,4} * a_{4,5} * a_{5,1}$ .

**Definition 1.18.** The PC matrix  $A$  on the graph  $\Gamma_I$  is consistent if and only if there exists a state  $s$  such that  $\mathcal{H}_s = \{e_G\}$ .

1.4.2. Ranking the trustworthiness of indirect comparisons

The main problem with hierarchy-less comparisons of two states  $s$  and  $s'$  is that paths of any length give comparison coefficients which cannot be distinguished. An indirect comparison, given by a path with 3 vertices, has the same status as a comparison involving a path with 100 vertices. This is why we need to introduce a grading on  $\mathcal{H}_{s,s'}$  called order. This terminology will be justified by the propositions thereafter.

**Definition 1.19.** Let  $s$  and  $s'$  be two states.

- Let  $\gamma$  be a path on  $\Gamma_I$  from  $s$  to  $s'$ . The length of  $\gamma$ , denoted by  $l(\gamma)$ , is the number of vertices of  $\gamma$ , and by  $H(\gamma)$  its holonomy.
- Let  $h \in \mathcal{H}_{s,s'}$ . The order of  $h$  is defined as

$$ord(h) = \min \{ l(\gamma) \mid H(\gamma) = h \}.$$

As a trivial consequence of the triangular equality, and as a justification of the terminology, we have:

**Proposition 1.20.** Let  $s, s'$  and  $s''$  be three states. Let  $(h, h') \in \mathcal{H}_{s,s'} \times \mathcal{H}_{s',s''}$ . Then

$$ord(hh') \leq ord(h) + ord(h').$$

Left action, right action and adjoint action of  $G^I$  extend straight way to PC matrices on  $\Gamma_I$  setting

$$\forall g \in G, \quad g * 0 = 0 * g = 0.$$

Adapting the proof of Theorem 1.7 we get:

**Theorem 1.21.** Let  $A = (a_{i,j})_{(i,j) \in I^2}$  be a PC matrix on  $\Gamma$ . Then  $A$  is consistent if and only if there exists  $(\lambda_i) \in G^I$  such that

$$a_{i,j} = \lambda_i^{-1} * \lambda_j$$

when  $a_{i,j} \neq 0$ .

Let  $A$  be a PC matrix on  $\Gamma_I$ . Inconsistency will be given here by the holonomy of a loop. Let us recall that the holonomy of a loop  $\langle s_{i_1}, s_{i_2}, \dots, s_{i_k}, s_{i_1} \rangle$  is trivial if and only if

$$a_{i_1,i_k} * (a_{i_1,i_2} * \dots * a_{i_{k-1},i_k})^{-1} = e_G.$$

This relation has to be compared with formula (2). The principle of ranking inconsistency with loop length gives the following:

**Definition 1.22.** Let  $\mathcal{F} : G \rightarrow \mathbb{R}_+$  be a map such that  $I(e_G) = 0$ . Let  $s$  be a base point on  $\Gamma_I$ . The **ranked Koczkodaj's inconsistency map** associated to  $\mathcal{F}$  the map

$$Kii_{\mathbb{N}} : s \mapsto Kii_{\mathbb{N}}(s) = \sum_{n \in \mathbb{N}} a_n X^n$$

where

$$a_n = \sup \{ \mathcal{F}(H(\gamma)) \mid \gamma \text{ is a loop at } s \text{ and } l(\gamma) = n \}.$$

One can easily see that  $a_n$  generalize  $Kii_n$ , and  $Kii_{\mathbb{N}}$  is a  $\mathbb{R}[[X]]$ -valued inconsistency map. Adapting Proposition 1.14, we get the following property:

**Proposition 1.23.**  $Kii_n$  is an Ad-invariant inconsistency map if and only if  $\mathcal{F}$  is Ad-invariant.

### 1.5. Minimizing inconsistency and the necessity of topological structures on $G$

#### 1.5.1. Distance matrix

In this section,  $G = \mathbb{R}_+^*$ . Setting

$$k_{i,j} = |\log(a_{i,j})|$$

we get another matrix, that we define as the *distance matrix*

$$K = (k_{i,j})_{(i,j) \in I^2}.$$

Notice that, if the coefficients of this matrix satisfy the triangle inequality  $\forall (i,j,l) \in I^3, k_{i,l} \leq k_{i,j} + k_{j,l}$ , we get a curvature matrix for metric spaces [35]. Due to the absolute value, we have the following:

**Proposition 1.24.** Let  $K$  be a non-zero distance matrix on  $\Delta_n$ . Let  $N$  be the number of non-zero coefficients in  $K$ . Then  $N$  is even and there exist  $2^{N/2}$  corresponding PC matrices.

**Outline of proof.** For each  $k_{i,j} \neq 0$ , we have that  $\log(a_{i,j}) = k_{i,j}$  or  $\log(a_{i,j}) = -k_{i,j}$ .  $\square$

Therefore, we have the following results:

**Proposition 1.25.** Let  $K$  be the distance matrix on  $\Delta_n$  associated to a consistent PC matrix  $A$ , which is assumed to be non-zero. Let  $N'$  be the number of coefficients  $k_{i,i+1}$  which are non-zero. Then there exists  $2^{N'}$  PC matrices built with the coefficients  $k_{i,i+1}$ .

**Proof.** The sign of  $\ln(a_{i,i+1})$  gives the  $2^{N'}$  consistent PC matrices which correspond to the coefficients  $k_{i,i+1}$ .  $\square$

So that, PC matrices cannot be encoded as distance matrices in their own generalities when  $G = \mathbb{R}_+^*$ . This suggests that the approach with  $G$ -distances suggested in [25] does not generalize PC matrices but only describes a different, quite similar tool.

#### 1.5.2. Vincinities, neighbourhoods of $CPC_I$ and inconsistency maps

Let us remark that for  $I = \mathbb{N}_3$ ,  $PC_3(G)$  splits by various ways into

$$CPC_3(G) \times G.$$

One map which realizes this one-to-one correspondence is

$$\begin{aligned} \Phi_3 : \left( \begin{array}{ccc} 1 & a_{1,2} & a_{1,3} \\ a_{2,1} & 1 & a_{2,3} \\ a_{3,1} & a_{3,2} & 1 \end{array} \right) \in PC_3(G) \\ \mapsto \left( \left( \begin{array}{ccc} 1 & a_{1,2} & a_{1,2}a_{2,3} \\ a_{2,1} & 1 & a_{2,3} \\ a_{3,2}a_{2,1} & a_{3,2} & 1 \end{array} \right), a_{1,2}a_{2,3}a_{3,1} \right) \in CPC_3(G) \times G. \end{aligned}$$

By the way,  $F_3(G)$  can be identified with  $G$ . The same way, one can define:

$$\Phi_n : PC_n(G) \rightarrow CPC_n(G) \times G^{\frac{(n-1)(n-2)}{2}}$$

by

$$\begin{aligned} \Phi_n \left( (a_{i,j})_{(i,j) \in \mathbb{N}_n^2} \right) \\ = \left( (b_{i,j})_{(i,j) \in \mathbb{N}_n^2}, a_{1,2}a_{2,3}a_{3,1}, a_{2,3}a_{3,4}a_{4,2}, \dots, a_{1,2}a_{2,3}a_{3,4}a_{4,1}, \dots \right), \end{aligned}$$

where the  $G$ -coefficients  $b_{i,j}$  are defined by:

- $b_{i,i} = 1_G$ ,
- $\forall i \in \mathbb{N}_{n-1}, b_{i,i+1} = a_{i,i+1}$  and  $b_{i+1,1} = b_{i,i+1}^{-1} = a_{i+1,i}$ ,
- $\forall i \in \mathbb{N}_{n-1}, \forall p \in \mathbb{N}_{n-1-i}, b_{i,i+1+p} = a_{i,i+1}a_{i+1,i+2} \dots a_{i+p,i+p+1}$  and  $b_{i+1+p,i} = b_{i,i+1+p}^{-1}$ ,

which defines a consistent PC matrix.

Notice that we recognize in the  $G^{\frac{(n-1)(n-2)}{2}}$ -component the coefficients present in the formulas for the generalization of Koczkodaj's inconsistency indicator given in Definition 1.22. Moreover, changing the PC matrix  $(a_{i,j})_{(i,j) \in \mathbb{N}_n^2}$  by Adjoint action of an element  $(g_i)_{i \in \mathbb{N}_n}$  of the gauge group  $G^n$  into  $(g_i \cdot a_{i,j} \cdot g_j^{-1})_{(i,j) \in \mathbb{N}_n^2}$  we get that:

- the matrix  $(b_{i,j})_{(i,j) \in \mathbb{N}_n^2}$  is also changed the same way into  $(g_i \cdot b_{i,j} \cdot g_j^{-1})_{(i,j) \in \mathbb{N}_n^2}$ ,
- the coefficients  $a_{i,i+1} \dots a_{i,i+k} a_{i+k,i}$  in the  $G^{\frac{(n-1)(n-2)}{2}}$ -component are changed by classical Adjoint action of  $G$  on itself, into  $g_i \cdot a_{i,i+1} \dots a_{i,i+k} a_{i+k,i} g_i^{-1}$ .

We turn now to inconsistency maps. Here the notion of inconsistency indicator is not necessary since we principally look at what happens when the value of the inconsistency map is at a neighbourhood of 0. An inconsistency map  $ii$  pulls back any topological filter in  $V_k$  into a filter in  $PC_n(G)$ . One of particular interest is the filter  $\mathcal{F}(V_k)$  of neighbourhoods of 0 in  $V_k$ .

**Definition 1.26.** The **fundamental filter** of an inconsistency map  $ii$  is the filter

$$ii^*(\mathcal{F}(V_k))$$

generated by

$$\{ ii^{-1}(A) \mid A \in \mathcal{F}(V_k) \}.$$

Let us precise that the map  $ii$  is not assumed continuous. Therefore, even if  $\mathcal{F}(V_k)$  is a filter of neighbourhoods, its pull-back  $ii^*(\mathcal{F}(V_k))$  needs not to be a filter of neighbourhoods, generated by a family of open sets, in any “reasonable” topology on  $PC_I(G)$ . This is the reason why we prefer to mention “filters” to “neighbourhoods” in our setting.

The following proposition is straightforward:

**Proposition 1.27.** *An inconsistency map  $ii$  on  $PC_n(G)$  is faithful if and only if*

$$\bigcap_{A \in \mathcal{F}(V_k)} ii^{-1}(A) = CPC_n(G).$$

Indeed the very wide variety of such filters around  $CPC_n(G)$ , even if inconsistency maps  $ii$  can be, in a first approach, assumed faithful and  $C^0$ -maps, shows that there is at this step no way to decide how an inconsistency map can be better than another. However, the notion of Ad-invariance of inconsistency maps can furnish a preferred restricted class. Gathering the results given before, we get that

$$F_n(G) \sim G^{\frac{(n-1)(n-2)}{2}}.$$

So that, one can generate faithful Ad-invariant inconsistency maps from functions  $f : F_n(G) \rightarrow V_k$  such that  $f^{-1}(0)$  is the orbit  $CPC_n(G)$ . In this case, orbits are level lines of  $ii = f \circ \pi$ . Koczkodaj’s inconsistency indicator is such an example.

1.6. Differential geometric methods on PC matrices

We now recall a technical but well-known and key result:

**Theorem 1.28 (Yamabe’s Theorem).** [36] *Any locally compact topological subgroup of an analytic Lie group is an analytic Lie group.*

Hence, from the natural topological properties derived from the previous section, we can assume with almost no loss of generality that  $G$  is a Lie group, at least for groups which have a presentation into matrix groups.

1.6.1. PC matrices read on a simplex

An exposition on holonomy is given in [28, 29, 37, 38]. The geometry of simplexes is well addressed by [26, 27], and the notion of Lie group is described in [37, 38]. Examples of finite dimensional Lie groups are provided by (classical) groups of (invertible) matrices, where multiplication and inversion are smooth coefficientwise. Other examples can be provided using a very general framework of differentiable manifolds, but most finite dimensional Lie groups can be realized as groups of matrices.

Let  $(G, \cdot)$  be a Lie group with Lie algebra  $(\mathfrak{g}, +, [\cdot, \cdot])$ . The expression “Lie group” is here understood in a very general sense. This can be a finite dimensional or an infinite dimensional group, or even a Frölicher group with Lie algebra [29, 39]. However, the beginner in the topic of Lie group is strongly advised to consider  $G$  as a matrix group. The only technical requirement for the sequel is the existence of an exponential map

$$\exp : C^\infty([0; 1], \mathfrak{g}) \rightarrow C^\infty([0; 1], G)$$

solving the logarithmic equation  $g^{-1}.dg = v$ , where  $v \in C^\infty([0; 1], \mathfrak{g})$  and  $g \in C^\infty([0; 1], G)$ . This ensures the existence of the holonomy of a connection [39]. Such a property is always fulfilled for finite dimensional groups, but not for Frölicher Lie groups. We get an example of Frölicher Lie group with no exponential map considering  $G = Diff_+(]0; 1[)$ , the group of increasing diffeomorphisms of the open unit interval [40].

On a trivial principal bundle  $P = M \times G$ , the horizontal lift of a path  $c \in C^\infty([0; 1], M)$  from a starting point  $p = (c(0), g_0) \in M \times G$  with respect to a connection  $\theta$  is the path  $\tilde{c} = (c, g) \in C^\infty([0; 1], P)$  such that

$$g^{-1}.dg = \theta(dc).$$

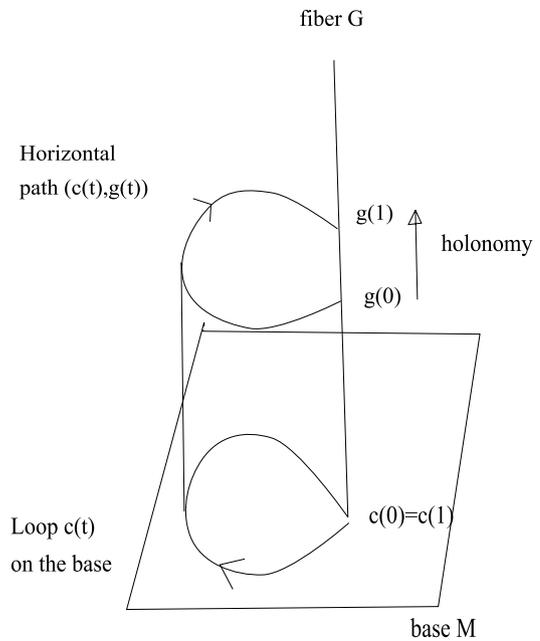


Fig. 2. Holonomy of a loop  $c(t)$ .

If  $c$  is a loop, we have  $Hol_{g(0)}c = g(0)^{-1}.g(1)$ . The holonomy of a loop depends on the base point  $(\gamma(0), g(0))$  and is invariant under co-adjoint action, see Fig. 2. In this text, the coadjoint action is understood as an action of  $G$  on the (total) space  $TG = G \times \mathfrak{g}$  in the spirit of [41], which allows the same notation for the co-adjoint action of  $G$  on itself or on its Lie algebra  $\mathfrak{g}$ .

Let  $n \in \mathbb{N}^*$  and

$$\Delta_n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \left( \sum_{i=0}^n x_i = 1 \right) \wedge (\forall i \in \{0, \dots, n\}, x_i \geq 0) \right\}$$

be an  $n$ -simplex. This simplex can be generalized to the infinite dimension:

$$\Delta_{\mathbb{N}} = \left\{ (x_n)_{n \in \mathbb{N}} \in l^1(\mathbb{N}, \mathbb{R}_+^*) \mid \sum_{i=0}^{\infty} x_i = 1 \right\}$$

and

$$\Delta_{\mathbb{Z}} = \left\{ (x_n)_{n \in \mathbb{Z}} \in l^1(\mathbb{Z}, \mathbb{R}_+^*) \mid \sum_{i \in \mathbb{Z}} x_i = 1 \right\},$$

where the summation over  $\mathbb{Z}$  is done by integration with respect to the counting measure. In the sequel,  $\Delta$  will denote  $\Delta_n, \Delta_{\mathbb{N}}$  or  $\Delta_{\mathbb{Z}}$ . Since  $\Delta$  is smoothly contractible, any  $G$ -principal bundle over  $\Delta$  is isomorphic to  $\Delta \times G$  and a  $G$ -connection 1-form on  $\Delta$  is a 1-form  $\theta \in \Omega^1(\Delta, \mathfrak{g})$ , which extends to a  $G$ -covariant 1-form in  $\Omega^1(\Delta, \mathfrak{g})$ , with respect to the coadjoint action of  $G$  on  $\mathfrak{g}$ . We define a gauge  $(g_i)_{i \in I} \in G^I$  with  $\tilde{\gamma}_i(1) = (\gamma_i(1), g_i)$  where

$$\gamma_i = \langle s_0, s_1 \rangle \vee \dots \vee \langle s_{i-1}, s_i \rangle \text{ if } i > 0$$

and

$$\gamma_i = \langle s_0, s_{-1} \rangle \vee \dots \vee \langle s_{i+1}, s_i \rangle \text{ if } i < 0.$$

We set  $g_i = Hol_{(s_0, 1_G)} \gamma_i$ . Let us recall that, for two paths  $c$  and  $c'$  such that  $c \vee c'$  exists (i.e.  $c(1) = c'(0)$ ), if  $p = (c(0), e_G)$ ,  $p' = (c'(0), e_G)$  and  $h = Hol_p c$ , we have:

$$Hol_p(c * c') = Hol_p(c).(h^{-1} Hol_{p'}(c')h) = Hol_{p'}(c').Hol_p(c). \tag{7}$$

Let

$$a_{i,j} = s_j \cdot Hol_{(s_0, e_G)} \left( \gamma_i[s_i, s_j] \gamma_j^{-1} \right) \cdot s_i^{-1}. \tag{8}$$

In the light of these specifications, we set

$$A = (a_{i,j})_{(i,j) \in I^2}$$

and the required notion of consistency is contravariant consistency.

**Proposition 1.29.** *A is a PC matrix.*

**Proof.** This follows from holonomy in “reverse orientation”.  $\square$

Let  $\gamma_{i,j,k} = \gamma_i * [s_i, s_j] * [s_j, s_k] * [s_k, s_i] * \gamma_i^{-1}$  be the loop based on  $s_i$  along the border of the oriented 2-vertex  $[s_i, s_j, s_k]$ , where  $*$  is the composition of paths. By Equation (7), contravariant consistency seems to fit naturally with flatness of connections:

$$\begin{aligned} \forall i, j, k, \quad a_{i,k} &= a_{j,k} \cdot a_{i,j} \\ \Leftrightarrow a_{k,i} \cdot a_{j,k} \cdot a_{i,j} &= a_{i,i} = 1_G \\ \Leftrightarrow Hol(\gamma_{i,j,k}) &= 1_G \end{aligned}$$

It is necessary to examine whether every PC matrix can be expressed as a matrix of holonomies of a fixed connection. For this, we need to assume that the group  $G$  is exponential, which means that the exponential map  $\mathfrak{g} \rightarrow G$  is onto.

**Proposition 1.30.** *If  $G$  is exponential, the map*

$$\Omega^1(\Delta, \mathfrak{g}) \rightarrow \{ \text{PC matrices} \}$$

$$\theta \mapsto \text{the holonomy matrix}$$

*is onto.*

**Proof.** Let  $A = (a_{i,j})_{(i,j) \in I^2}$  be a PC matrix. Let us build a connection 1-form  $\theta \in \Omega^1(\Delta, \mathfrak{g})$  such as (8). For this, before constructing our connection, we fix the gauge  $(g_i)_{i \in I} \in G^I$  by

$$g_i = a_{0,1} \dots a_{i-1,i} \text{ for } i > 0$$

and by

$$g_i = a_{0,-1} \dots a_{i+1,i} \text{ for } i < 0.$$

Once the gauge is fixed, we begin by dealing with each 1-vertex and use gauge covariance to extend the 1-form in  $\Omega^1(\Delta, \mathfrak{g})$  constructed to a ( $G$ -equivariant) connection 1-form on  $\Delta \times G$ , following [42].

Firstly, by fixing indexes  $i < j$ , which holds in particular for  $j = i + 1$ , we choose  $v_{i,j} \in \mathfrak{g}$  such that  $\exp(v_{i,j}) = a_{i,j}$ . Needless to say, the condition  $v_{j,i} = -v_{i,j}$  is consistent with  $a_{i,j}^{-1} = a_{j,i}$ . The group  $\{\exp(tv_{i,j}) \mid t \in \mathbb{R}\}$  is an abelian subgroup of  $G$ . For this reason, formulas for holonomy on an abelian group can be used to specify a function  $f_{i,j} : [0, 1] \rightarrow \mathbb{R}_+ \cdot v_{i,j}$ , with support in  $[1/3; 2/3]$ , and such that  $\int_0^1 f_{i,j}(s) ds = v_i$  on the length-parametrized edge  $\langle s_i, s_j \rangle$ . Finding such a function is possible, and extending the  $G$ -equivariant 1-form  $f_{i,j} ds$  on  $[s_i, s_j] \times G$  to a  $G$ -equivariant 1-form  $\theta_{i,j}$  on  $\Delta \times G$  which is null off  $V_{i,j} \times G$ , where  $V_{i,j}$  is a tubular neighbourhood of radius  $\epsilon > 0$  of  $\text{supp}(f_{i,j})$ , is also possible.

Secondly, we repeat this procedure for each couple of indexes  $(i, j)$  such that  $i < j$ , and choose  $\epsilon$  small enough in order to have non-intersecting supports  $\text{supp}(\theta_{i,j})$ , for example  $\epsilon = 1/6$ . By setting

$$\theta = \sum_{i < j} \theta_{i,j},$$

we get a connection  $\theta$  whose holonomy matrix is given by  $A$ .  $\square$

Let us provide a geometric criterion for consistency.

**Proposition 1.31.** *If the connection  $\theta$  is flat,  $A$  is a contravariant consistent PC matrix.*

**Proof.**  $\theta$  is flat if and only if its curvature is null. This implies that the Lie algebra of the holonomy group is null, and since each 2-vertex  $[s_i, s_j, s_k]$  is contractible, the holonomy group is trivial.  $\square$

### 1.6.2. The gauge group acting on the space of contravariant PC matrices

The gauge group is defined before, and we give the following easy result which justifies the terminology:

**Theorem 1.32.** *The coadjoint map*

$$Ad : G^I \times \{ \text{contravariant PC matrices} \} \rightarrow \{ \text{contravariant PC matrices} \}$$

$$(\tilde{g}_i)_{i \in I} \times (a_{j,k})_{(j,k) \in I^2} \mapsto (\tilde{g}_k^{-1} a_{j,k} \tilde{g}_j)_{(j,k) \in I^2}$$

*transforms consistent PC matrices to consistent PC matrices. Moreover, for each  $(\tilde{g}_i)_{i \in I} \in G^I$ , the map  $Ad_{(\tilde{g}_i)_{i \in I}}$  is one-to-one and onto.*

Let us now recall some basics on the gauge group on  $\Delta \times G$ . This group is given by  $C^\infty(\Delta, G)$  and acts on the space of connections  $\Omega^1(\Delta, G)$  by the formula:

$$(g, \theta) \in C^\infty(\Delta, G) \times \Omega^1(\Delta, G) \mapsto g^{-1} dg + Ad_{g^{-1}} \theta.$$

Under these conditions, the holonomy  $Hol_p(\gamma)$  of a loop  $\gamma$  transforms into  $Ad_{g^{-1}}(Hol_p(\gamma))$  at the same base point  $p$  of the principal bundle  $\Delta \times G$ .

**Theorem 1.33.** *If  $G$  is a compact exponential finite dimensional Lie group, then for each  $(\tilde{g}_i)_{i \in I} \in G^I$ , there exists  $g \in C^\infty(\Delta, G)$  such that  $g(s_i) = \tilde{g}_i$ . Moreover, for any contravariant PC matrix  $A$  which is the holonomy matrix of a connection  $\theta$ , then  $Ad_{(\tilde{g}_i)_{i \in I}}(A)$  is the holonomy matrix of the connection  $g^{-1} dg + Ad_{g^{-1}} \theta$ .*

**Proof.** Since  $G$  is exponential, setting  $\text{cut}(G)$  to the cut-locus of  $G$  with respect to the exponential map, we have that  $G - \text{cut}(G)$  is star-shaped. So, there exists a continuous map  $g : \Delta \rightarrow G$  which restricts to  $\Delta - \partial\Delta \rightarrow G - \text{cut}(G)$  such that  $g(s_i) = \tilde{g}_i$ . Since  $G$  is compact (and hence finite-dimensional), this map can be chosen smooth [43]. Once this map is constructed, the rest of the theorem follows from classical properties of the gauge group action that we have sketched before.  $\square$

### 1.6.3. When $G$ is a free abelian Lie group

Assume that  $G = (\mathbb{R}_+^*)^J$  where  $J$  has any cardinality, finite or infinite. In this case,

$$Hol(\langle s_i, s_j \rangle) = e^{\int_0^1 \theta(ds_{i,j})}$$

where  $ds$  is the unit vector of the normalized length parametrization of  $[s_i, s_j]$ . Thus,

$$Hol(\gamma_{i,j,k}) = 1_G \Leftrightarrow \int_0^1 \theta(ds_{i,j}) + \int_0^1 \theta(ds_{j,k}) + \int_0^1 \theta(ds_{k,i}) = 0 \tag{9}$$

The connection  $\theta$  is flat now and it reads as  $d\theta = 0$  which is equivalent to  $\theta = df$ , where  $f \in C^\infty(\Delta, \mathbb{R}^J)$  (because  $H^1(\Delta, \mathbb{R}) = 0$ ). With this function  $f$ , setting  $e^{f(s_i)} = \lambda_i$ , we recover the “basic consistency condition” of Theorem 1.6. In the spirit of Whitney’s simplicial approximation [26], we assume that  $G = \mathbb{R}_+^*$  and  $\mathfrak{g} = \mathbb{R}$  for simplicity, and our computations will extend to  $\mathbb{R}^J$  component-wise. Let us construct an affine function  $f$ . This function is uniquely determined by its values  $f(s_i)$ , for  $i \in \{0, \dots, n\}$  and we get the system:

$$\begin{cases} f(s_0) - f(s_1) & = -\ln(a_{0,1}) \\ f(s_1) - f(s_2) & = -\ln(a_{1,2}) \\ & \dots \\ f(s_{n-1}) - f(s_n) & = -\ln(a_{n-1,n}) \end{cases} \tag{10}$$

which is a  $n$ -system with  $(n + 1)$  variables. Since  $\theta = df$ , we can normalize it, assuming e.g.  $f(s_0) = 0$ , and the system gets a unique solution, and hence a unique affine function  $f$  and a unique connection  $\theta = df$ . Now, setting  $\lambda_i = e^{f(s_i)}$ , we recover the construction given in the proof of Theorem 1.6 for this particular choice of group  $G$ .

1.6.4. Remarks on methods for minimizing inconsistency

Let us fix the set  $PC_n(G)$  and an inconsistency indicator  $ii$ . We assume here that the chosen inconsistency indicator  $ii$  is  $C^0$ ,  $\mathbb{R}_+$ -valued and of  $C^k$ -regularity off the set  $CPC_n(G)$ , with  $k$  large enough. Passing from  $G = \mathbb{R}_+^*$  to a general Lie group  $G$ , most techniques used in the existing scientific literature on pairwise comparisons cannot be applied straight way because they are based on three properties of  $\mathbb{R}_+^*$  which are still not valid on a Lie group  $G$ , namely:

- (1)  $\mathbb{R}_+^*$  is contractible;
- (2)  $\forall x \in \mathbb{R}_+^*, \exists ! y \in \mathbb{R}_+^*, y^n = x$ ;
- (3)  $\mathbb{R}_+^*$  is abelian.

However, the inconsistency indicator  $ii$  appears like a functional which has to be minimized. This kind of problem has been approached by many ways in optimization techniques, where enough regularity of the functional is needed. We leave these questions for future works, because the framework of application may impose to choose an optimization scheme or another. We also have to highlight that this problem of minimization for functional has a pending problem in the field of mathematical physics. In particular, when  $ii$  is Koczkodaj's inconsistency indicator, minimizing

$$ii(A) = \sup\{ii_3(B) \mid B \subset A\}$$

is similar to minimizing the holonomy along boundaries of 2-simplexes. This approach leads to consider Yang-Mills type functionals along the lines of [19, 42], which will be studied extensively in a future work.

1.7. Uncertainty and precision of measurements in pairwise comparisons

We now turn to probabilistic aspects. For this we must assume, a priori, that there exists a **bi-invariant Radon measure**  $\lambda$  on the topological group  $G$ , which is also called **Haar measure**. For such a measure, for any integrable function  $f : G \rightarrow \mathbb{R}$ , we have the bi-invariance properties of integration with respect to  $\lambda$ :

$$\forall g \in G, \int_G f(x)d\lambda(x) = \int_G f(g \cdot x)d\lambda(x) = \int_G f(x \cdot g)d\lambda(x).$$

For example,

- (1) for  $n \in \mathbb{N}^*$ , (classical) Riemannian integration of functions  $\mathbb{R}^n \rightarrow \mathbb{R}$  involve the Haar measure of the abelian group  $(\mathbb{R}^n, +)$  called Lebesgue measure.
- (2) The multiplicative group  $(\mathbb{R}_+^*, \cdot)$  has a Haar measure different from the Lebesgue measure on  $(\mathbb{R}, +)$ , because the group law is different. The exponential map

$$exp : \mathbb{R} \rightarrow \mathbb{R}_+^*$$

is an isomorphism of topological groups. Setting  $\lambda$  and  $\lambda'$  respectively the Lebesgue measure on  $\mathbb{R}$  and the Haar measure on  $\mathbb{R}_+^*$ , if  $g : \mathbb{R}_+^* \rightarrow \mathbb{R}$  is a  $C^0$ -function and  $f = g \circ exp$ , we have

$$\int_{\mathbb{R}} f d\lambda = \int_{\mathbb{R}_+^*} g d\lambda' = \int_0^{+\infty} \frac{g(x)}{x} dx.$$

- (3) If  $G$  is compact, the Haar measure has finite volume and can be normalized into a probability measure.

On one hand, considering only the Haar measure would mean that the measurements are totally random, without any consideration on the real situation under evaluation. On the other hand, exact evaluations belong to an idealist picture, and errors can occur from many different ways: measurements, expert inability among others. Moreover, error perception is limited to a fixed precision. These are the reasons why a measurement is better modeled by a probability distribution which “concentrates” around its expectation value which is chosen as the exact value of the measurement. Among the most known examples, Gaussian measures play a very important role. On  $\mathbb{R}$ , a Gaussian probability measure  $\gamma$  such that  $\mathbb{E}(\gamma) = 0$  reads as

$$\gamma(A) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{-\frac{x^2}{2\sigma^2}} d\lambda. \tag{11}$$

When  $\sigma \rightarrow 0^+$ ,  $\gamma$  converges, for the vague topology of probability measures, to the Dirac measure  $\delta_0$  defined by  $\delta_0(A) = 1$  if  $0 \in A$ , and  $\delta_0(A) = 0$  if  $0 \notin A$ . In this picture, 0 reads as the exact measure, and  $\mu(A)$  reflects with which probability the measurement obtained is in the set  $A \subset \mathbb{R}$ .

We propose here two ways of generalization of this picture to pairwise comparisons matrices. In these two approaches, we assume that each coefficient  $a_{i,j}$  is a random variable.

- On the one hand, let us consider a measure  $\gamma_{i,j}$  on  $G$  which models, for any measurable set  $U \subset G$ , the probability to get  $a_{i,j} \in U$  after “expert evaluation”. Identifying  $PC_n(G)$  with  $G^{\frac{n(n-1)}{2}}$  we get a product measure  $\otimes_{1 \leq i < j \leq n} \gamma_{i,j}$  on  $PC_n(G)$ . Thus for any inconsistency map

$$ii : PC_n(G) \rightarrow \mathbb{R}_+,$$

assuming that inconsistency of  $A \in PC_n(G)$  is acceptable if  $ii(A) < \epsilon$  for a fixed value  $\epsilon > 0$ , the probability for the acceptable inconsistency is given by

$$\otimes_{1 \leq i < j \leq n} \gamma_{i,j} \left( \{A \in PC_n(G) \mid ii(A) \leq \epsilon\} \right).$$

- On the other hand, let us assume that the chosen inconsistency map  $ii$  reflects the global perception of the inconsistency of the system of an extra-observer, whose natural tendency will lead him to minimize inconsistency, ideally at the level  $ii < \epsilon$ . Then we can propose a Feynman-Kac type formula by introducing the (normalized) measure  $\mu_{ii}$  of density  $e^{-\frac{ii(A)}{\epsilon}}$  with respect to the Haar probability measure  $\lambda$  on  $G^{\frac{n(n-1)}{2}} \sim PC_n(G)$  given by the formula:

$$\mu(A) = \frac{1}{Z} \int_A e^{-\frac{ii(x)}{\epsilon}} d\lambda(x)$$

with

$$Z = \int_{PC_n(G)} e^{-\frac{ii(x)}{\epsilon}} d\lambda(x).$$

1.8. Examples

1.8.1.  $GL_n$ -comparisons

Let  $S = \mathbb{R}^\infty$  be the inductive limit of the family  $\{\mathbb{R}^n; n \in \mathbb{N}^*\}$  such that the inclusions  $\mathbb{R}^n \subset \mathbb{R}^{n+1}$  is the canonical inclusion with respect to the first coordinates. Here,  $S$  is an object of the category of vector spaces. With this setting, we get  $G = GL_\infty(\mathbb{R})$ , which is the inductive limit of the family  $\{GL_n(\mathbb{R}); n \in \mathbb{N}^*\}$ . If  $I$  is a finite set of indexes (e.g.  $I = \mathbb{N}_k$  for some  $k \in \mathbb{N}^*$ ),  $\sup_{i \in I} dim(s_i) < +\infty$  and setting  $n = \sup_{i \in I} dim(s_i)$ , we work with the restricted setting  $S = \mathbb{R}^n$  and  $G = GL_n(\mathbb{R})$ .

**Remark 1.34.** Even if there exists an index  $i \in I$  such that  $n_i < n$ , we have to consider the inclusion  $\mathbb{R}^{n_i} \subset \mathbb{R}^n$  because there is no linear isomorphism from  $\mathbb{R}^{n_i}$  to  $\mathbb{R}^n$  (by the theorem of dimension).

With this construction, we get a first family of inconsistency maps. The determinant map

$$\det : GL_n(\mathbb{R}) \rightarrow \mathbb{R}^*$$

is a group morphism.

**Proposition 1.35.** Let  $ii_{\mathbb{R}_+^*}$  be a  $(\mathbb{R}_+^*$ -valued) inconsistency map on  $PC_I(\mathbb{R}_+^*)$ . Then the map

$$ii_{det} : PC_I(GL_n(\mathbb{R})) \rightarrow \mathbb{R}_+^*$$

$$A = (a_{i,j})_{i \in I} \mapsto ii_{\mathbb{R}_+^*} ( (|\det(a_{i,j})|)_{(i,j) \in I^2} )$$

defined as a composed map

$$ii_{det} = ii_{\mathbb{R}_+^*} \circ |\det(\cdot)|,$$

is a non-faithful, Ad-invariant inconsistency operator on  $PC_I(GL_n(\mathbb{R}))$ .

**Proof.** The only non-trivial part is non-faithfulness. For this, let us give a counter-example. Let  $A \in PC_3(GL_2(\mathbb{R}))$  defined by  $a_{1,2} = a_{2,3} = -a_{1,3} = I_2$ . Then

$$a_{1,2}a_{2,3} \neq a_{1,3}$$

where as

$$|\det(A)| = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in CPC_3(\mathbb{R}_+^*),$$

thus  $ii_{det}(A) = 0$ .

We moreover remark that this counter-example on  $3 \times 3$ -PC matrices can be used to produce other counter-examples for  $n \times n$ -PC matrices, by inserting the matrix  $A$  as a block of a PC matrix  $B \in PC_n(G)$ , for which all the other coefficients are equal to  $I_2$ .  $\square$

But this class of inconsistency maps is not the only one of interest, even if  $\det$  generates  $Hom(GL_n(\mathbb{R}), \mathbb{R}^*)$  in the category of groups. This example shows how situations expressed by a (more) complex group such as  $GL_n(\mathbb{R})$  cannot be reduced straight way (one would say naively) to a more simple abelian group such as  $\mathbb{R}_+^*$ .

### 1.8.2. Error in cartography and in tunnel building

Let us now describe two situations where non-abelian groups rise naturally in a description of errors by the pairwise comparisons method. For these two examples, the group under consideration is a group of orientation preserving, isometric affine transformations  $Is_n$  of a (finite dimensional) affine space  $\mathbb{R}^n$ .

**Example: cartography in a forest.** One of the main features in cartography, or during the recovery of an exit path, in a forest (here in a flat land) is the lack of external point where to get a precise indication on the actual position of the observer. By the way, moves and direction changes can only be appreciated by self-evaluation, which is subject to numerous, non-compensative errors in the appreciation of the positions during the path. More precisely,

- (1) moves along a straight line can be evaluated by a translation, i.e. a vector in  $\mathbb{R}^2$ ,
- (2) changes of direction can be evaluated as rotations, centred at the position of the observer.

Gathering these two aspects, a path in the forest can be assimilated to succession of moves transcribed by elements  $a_{1,2}, \dots, a_{n-1,n}$  of the group generated by planar rotations and translations. This group is the group of orientation preserving, isometric affine maps in  $\mathbb{R}^2$ . Thus if one can evaluate the  $n$ -th position with respect to the initial one, this gives another element  $a_{1,n}$ . Due to successive errors, very often

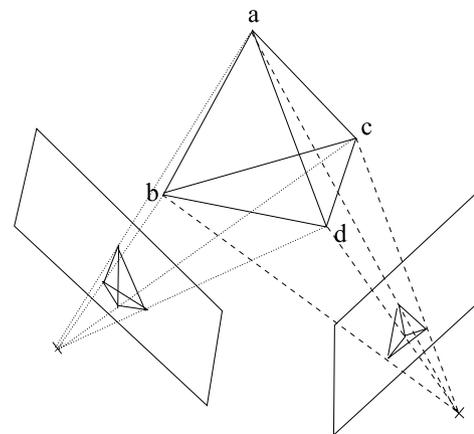


Fig. 3. Two projections of  $[abcd]$ .

$$a_{1,n} \neq a_{1,2}a_{2,3} \dots a_{n-1,n}.$$

This is exactly a situation of inconsistency where coefficients are in the non-abelian group  $Is_2$ .

**Example: error in tunnel building.** The situation is the same in tunnel building, where the surveyors need to indicate, at each step of perforation of a tunnel, in which direction one has to correct the next perforation step underground. Each tunnel starting from each side must meet exactly at the end of the process. For the same reasons as in previous example, the (non-abelian) group under consideration here is  $Is_3$ . Currently, the admissible error is in the range of 1 cm per 100 m of tunnel. This error is admissible inconsistency.

### 1.8.3. Perspective in image processing

Let  $[abcd]$  be a 3 simplex (tetrahedron). Let  $\omega \in \mathbb{R}^3 \setminus [abcd]$ , and let  $P_\omega$  be a (projection) plan, such that  $\omega \notin P_\omega$ . In projective perspective, the projection of  $x \in \mathbb{R}^3 - \{\omega\}$ , is  $x_0 \in P_\omega$  such that

$$\{x_0\} = (\omega, x) \cap P_\omega \quad (\text{if it exists}).$$

Let us recall that projections exist in a “generic” way, that is,  $x_0$  exists unless  $(x, \omega)$  and  $P_\omega$  are parallel. We also assume that  $\omega$  is “far enough” (at the “optical infinity”) so that, in first approximation, projections can be assimilated to affine projections on  $P_\omega$ .

Let us consider for simplicity, first, a tetrahedron  $[abcd] \subset \mathbb{R}^3$ , and let  $a_0, b_0, c_0$  and  $d_0$  the corresponding projections with respect to  $\omega$ . Any 2-simplex  $[xyz]$  of  $[abcd]$  projects to a 2-simplex  $[x_0y_0z_0]$  in  $P_\omega$ . For another choice  $(\omega', P_{\omega'})$ , we get other projections  $[x'_0y'_0z'_0]$  of  $[xyz]$ . See Fig. 3. Now, identifying  $P_\omega$  and  $P_{\omega'}$  with the standard plan  $\mathbb{R}^2$  by an arbitrary choice of coordinates, we give the numbers 1, 2, 3 and 4 to the faces resp.  $[abc]$ ,  $[abd]$ ,  $[acd]$  and  $[bcd]$  and, in  $\mathbb{R}^2$ , define  $a_{i,j}$  as the unique affine map  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  which transforms the  $i$ -th face to the  $j$ -th face for  $\omega$ -the projection, and  $a'_{i,j}$  the corresponding coefficient for the  $\omega'$ -projection. Let us now consider the unique affine map  $\lambda_i$  which transforms the  $i$ -th face from the  $\omega$ -projection to the  $\omega'$ -projection. We then have that

$$a_{i,j} = \lambda_j^{-1} a'_{i,j} \lambda_i$$

for any index, and the PC matrices  $(a_{i,j})$  and  $(a'_{i,j})$  are consistent, but the tetrahedron is flat if and only if

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda.$$

Thus, this leads to the following definition:

**Definition 1.36.** Let  $G$  be the group of affine bijections of  $\mathbb{R}^2$ . We call **perspective matrix** of  $[abcd]$  with respect to  $(\omega, P_\omega)$  and  $(\omega', P_{\omega'})$  the matrix  $(b_{i,j}) \in PC_4(G)$  defined by

$$b_{i,j} = \lambda_i a_{i,j} \lambda_i^{-1},$$

which is defined up to Ad-action.

Then we get the trivial proposition, passing through the consistency in  $PC_4(G)$ :

**Proposition 1.37.** *A matrix  $(b_{i,j}) \in PC_n(G)$  encodes a triangulated simplicial complex in  $\mathbb{R}^3$  if and only if it is in  $CPC_n(G)$ .*

Thus perspective of a 3-simplex can be encoded into a  $CPC_4(G)$  matrix.

## 2. Conclusions

Beyond the classical setting of pairwise comparisons matrices with coefficients in  $\mathbb{R}_+^*$  (which has a huge impact in any field where decision is crucial, see e.g. [44] for a comprehensive review of the main features), the need of pairwise comparisons with more general coefficients has been felt by various authors last years [1, 17, 21, 25, 45, 46] while non-numerical ranking was explored by other authors [12, 13]. Our approach is different. All these works are based on tentatives of analyzing possible extensions of the properties of pairwise comparisons matrices with coefficients in  $\mathbb{R}_+^*$ . Our work is based on the analogy with quantum physics which structures appear also in models of exchange markets. These models are also known as fields of application of pairwise comparisons and inconsistency indicators. The link between these three fields has been discussed briefly in [19] while these notes were written.

This work intends to provide a safe panel of mathematical properties on which operations research can be based, for any framework where pairwise comparisons in a group arise. We have tried to give a panoramic map of some directions in which a model of decision making by pairwise comparisons can be efficient, from the headland of actual ideas in quantum physics.

It appears to us that the application of the methods of quantum physics to pairwise comparisons can give rise to original, unexpected quantities for decision making, mostly because these fields have been developed totally independently till nowadays. Ideas coming from Yang-Mills theory, Maxwell equations, integrable systems and potential theory among others appear to us as promising in the context of operations research.

Conversely, a new interpretation of Heisenberg principle and quantification of space-time may be found by applying the formalism of pairwise comparisons, as well as new lights on electromagnetic fields, especially non-trivial magnetic potentials.

Due to their common mathematical structures, these two fields of research may enrich each other in a very next future.

## Declarations

### Author contribution statement

Jean-Pierre Magnot: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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## Additional information

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