Ultrasonics 92 (2019) 50–56

Contents lists available at ScienceDirect

Ultrasonics

journal homepage: www.elsevier.com/locate/ultras

Comparison of quasi-Rayleigh waves and Rayleigh waves, and clarifying the cut-off frequency of quasi-Rayleigh waves

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ABSTRACT

The Partial Wave Method is unique in that it establishes a foundation on which various elastodynamic guided waves can be compared. In this paper, the method is used to compare quasi-Rayleigh waves and Rayleigh waves, and investigate the eccentricities of the Partial Wave Method at phase velocities equal to the Rayleigh wave speed. The comparison results in the definition of two types of quasi-Rayleigh waves and an explanation for quasi-Rayleigh wave behavior reported in the literature at frequencies that do not satisfy Viktorov's quasi-Rayleigh wave condition. These conclusions are also verified by the superposition of A0 and S0 mode wave-structures calculated using the semi-analytical finite element method.

1. Introduction

To the best of the authors' knowledge, quasi-Rayleigh waves were first investigated in detail by Viktorov in [1], which was written in Russian and then translated into English and used as a subsection in Viktorov's book [2]. The contents of the book's subsection will be used primarily for this paper as opposed to the original paper. The quasi-Rayleigh wave is described as a superposition of A0 and S0 Lamb wave modes which interfere in such a way that the resulting wave-structure has an appearance similar to a Rayleigh wave. Michal et al. [3] contains a nice graphic showing the superposition and the resulting quasi-Rayleigh wave's wave-structure.

The quasi-Rayleigh wave will also exhibit a beat phenomenon in which the wave energy will appear to switch, back and forth, between the top and bottom surfaces of the plate. The effect of the beat phenomenon on wave-propagation can be seen in a waterfall plot shown by Masserey and Fromme [4]. Referred to as the beat length, L is the distance of propagation before the quasi-Rayleigh wave switches from one surface to the other and then back,

$$L = \frac{2\pi}{(k_{A0}-k_{S0})}. \quad (1)$$

The expression is intended to be a calculation at a single frequency where k_{A0} and k_{S0} are the wavenumbers of the A0 and S0 modes, respectively. The difference between wavenumbers yields a phase difference when multiplied by the propagation distance. When the phase difference is equal to π, the quasi-Rayleigh wave switches to the opposite surface. When the phase difference is equal to 2π, the quasi-Rayleigh wave switches back to its original surface [5]. Besides these characteristics, a general condition for the existence of a quasi-Rayleigh wave was approximated by Viktorov to be,

$$H \geq 2\lambda_R, \quad (2)$$

where H is the plate thickness and λ_R is the Rayleigh wave's wavelength [2]. Experimental evidence appears to be the origin of the condition, but the extent of the research is unknown from the literature available. For the purposes of this paper, henceforth, Eq. (2) will be referred to as Viktorov's condition. The term "frequency-thickness product" will be used frequently throughout the paper to refer to an axis-specific convention common to the ultrasonic guided wave literature, see Figure 8-8 in Rose's book [6].

There are two objectives to this paper. The first objective is to briefly address the relationship between the quasi-Rayleigh wave and the Rayleigh wave by using the Partial Wave Method. The second objective is to develop an explanation for why quasi-Rayleigh wave behavior has been experimentally observed at frequencies that do not satisfy Viktorov's condition. To meet these objectives, two types of quasi-Rayleigh waves will be defined based on the criteria stated in the literature; a so-called "strict" definition and a "lenient" definition.

To summarize the contents of the paper by section, Section 2 develops the case that quasi-Rayleigh waves and Rayleigh waves are indeed separate modes of wave-propagation by analyzing their relationship through the perspective of the Partial Wave Method. Section 3.1 discusses a quirk of the Partial Wave Method when using the approach outlined by Hakoda and Lissenden [7]. Sections 3.2 and 3.3 describe the strict and lenient definitions of a quasi-Rayleigh wave, respectively. The superposition of wave-structures calculated using the Semi-Analytical Finite Element (SAFE) method are provided to assist in verifying

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https://doi.org/10.1016/j.ultras.2018.08.021
Received 29 June 2018; Received in revised form 22 August 2018; Accepted 31 August 2018
Available online 01 September 2018

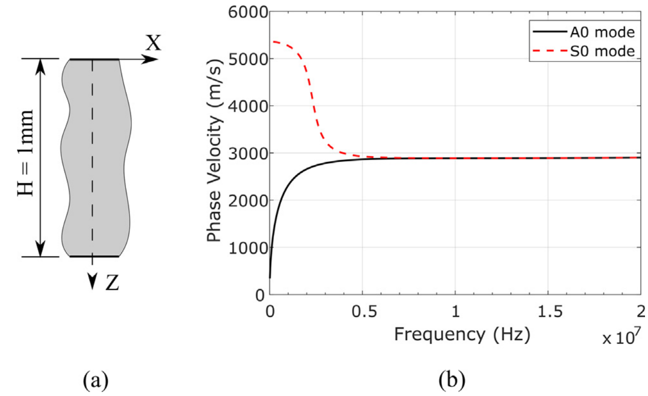

Fig. 1. (a) Schematic of assumed aluminum plate waveguide. (b) The S0 and A0 mode dispersion curves for a 1-mm thick aluminum plate.

the results of the Partial Wave Method analysis. All dispersion curves and wave-structures shown in this paper were calculated using the SAFE method implemented in COMSOL Multiphysics [8].

As in previous work [7], the eigenvectors for the partial waves were normalized before being used for the calculations shown in this paper. For all calculations a 1-mm thick aluminum waveguide will be used, the coordinate system, parameters and relevant dispersion curves are shown in Fig. 1.

2. Comparing the rayleigh wave and quasi-rayleigh wave

A Rayleigh wave is a wave which propagates along the surface of an elastic half-space. The wave decays exponentially with depth into the half-space and exhibits elliptical particle motion. The conventional derivation of the Rayleigh wave solution involves the use of potentials to determine a characteristic equation for a wave propagating on the surface of a theoretical half-space [2]. When considered for experimental applications, the question becomes "is this medium thick enough to approximate the half-space?" The existence of the quasi-Rayleigh wave answers this question by demonstrating how a wave with characteristics similar to a Rayleigh wave can propagate within a medium of finite thickness. However, the answer is incomplete since the relationship between the two is still unanswered. If only the characteristics of the two waves were compared, one could conjecture that all Rayleigh waves should be considered quasi-Rayleigh waves, since an ideal half-space is not physically realizable. The lack of a beat phenomenon in the conventional Rayleigh wave can even be explained by stating that at high frequency-thickness products, the A0 and S0 wavenumbers converge, which leads L in Eq. (1) to go to infinity. However, this argument could be considered sophistry by some. Instead, if we compare the two types of waves by using the Partial Wave Method as described by Hakoda and Lissenden [7], the relationship between the two becomes much clearer.

As shown in Hakoda and Lissenden [7], the Rayleigh wave can be constructed as a superposition of a longitudinal (L) surface partial-wave and a shear-vertical (SV) surface partial wave. That is, only two partial waves are required to accurately represent the Rayleigh wave solution. However, the quasi-Rayleigh wave requires eight partial waves to be accurately represented. That is, the quasi-Rayleigh wave is the superposition of the A0 and S0 modes, and each mode requires four partial waves to represent it. All of these waves though, the Rayleigh, A0, and S0 waves, can be constructed from approximately the same, fundamental partial waves as demonstrated previously [7]. That is, the matter of whether a Rayleigh wave or quasi-Rayleigh wave exists in the medium is a question of excitation and mode conversion. If only two partial waves are excited and the medium is thick enough such that it does not excite a surface wave on the opposite surface then it will remain a Rayleigh wave; but if the medium is thin, then it is possible to excite A0 and S0 modes and therefore a quasi-Rayleigh wave. If the finite-thickness medium were to suddenly change to a half-space, we can expect a quasi-Rayleigh wave (at high frequency-thickness) to mode-convert to a Rayleigh wave.

3. The partial wave method and the quasi-rayleigh wave

To reiterate, the fundamental characteristics of a quasi-Rayleigh wave are:

- Condition (1): the wave-structure looks similar to a Rayleigh wave's wave-structure,
- Condition (2): it exhibits the beat phenomenon.

These conditions will be referenced frequently in this section as we attempt to define frequency ranges in which a quasi-Rayleigh wave can exist.

3.1. The quirk of the Partial Wave Method framework

The Partial Wave Method, as applied in Hakoda and Lissenden [7], uses the superposition of partial waves to determine the wave-structure of Lamb waves. In particular, the composition of partial waves for each Lamb wave was calculated using three of the four available traction-free boundary conditions (BCs), while one of the partial wave amplitudes was assumed to be unity. The three boundary conditions that are used in this paper are the same as in Hakoda and Lissenden [7],

$$\sigma_{zz} = 0 \quad \text{at} \quad z = 0, \tag{3}$$

$$\sigma_{xz} = 0 \quad \text{at} \quad z = 0, \tag{4}$$

$$\sigma_{xz} = 0 \quad \text{at} \quad z = H. \tag{5}$$

This approach yields accurate wave-structure and partial wave composition for Lamb waves, except at the Rayleigh wave speed. This is easily noticeable at high frequency-thickness products, since the partial wave composition begins to become lopsided as the phase velocity converges to the Rayleigh wave speed. This becomes clear when plotting the phase variable, q, as a function of frequency as shown in Fig. 2, where q is the ratio between the partial wave amplitude on the top of the plate and the partial wave amplitude on the bottom of the plate. Going by the results in [7], q should equal -1 for the S0 mode and $+1$ for the A0 mode. From Fig. 2, it appears that the partial wave amplitudes begin to deviate from this at approximately 7.5 MHz. To be certain of the failure of the method at the Rayleigh wave speed for a high

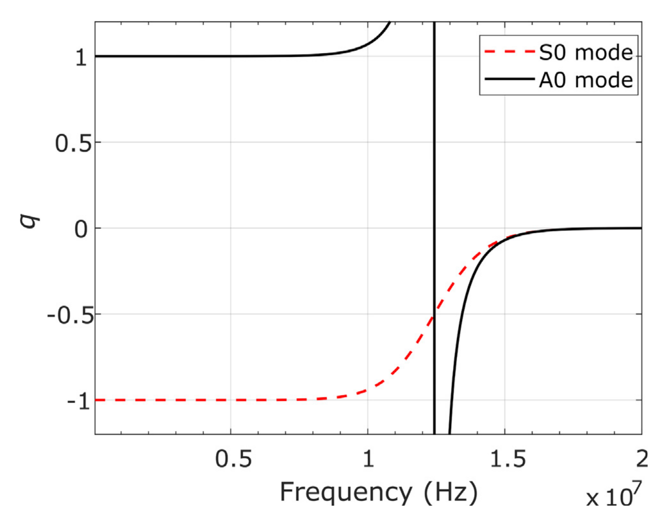

Fig. 2. The phase variable, q, which is the ratio of the partial wave amplitude on the plate top surface to the amplitude on the bottom surface. The plot shows q for the S0 and A0 Lamb wave modes with respect to frequency.

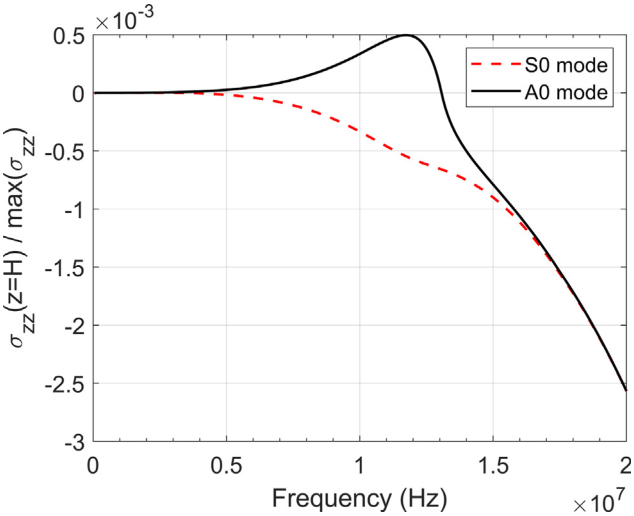

Fig. 3. Normalized σ_{zz} stress at $z = H$. For a given mode and frequency the stress was normalized with respect to the maximum σ_{zz} present in plate.

frequency-thickness product, the stress not used in the BCs, $\sigma_{zz}(z = H)$, can be plotted with respect to frequency to see when it stops being equal to zero. This is shown in Fig. 3 where the stress is normalized by the maximum stress through the thickness of the plate for that mode at the given frequency. This is done for both the A0 and S0 modes. σ_{zz} at $z = H$ begins to noticeably increase in magnitude at around 5 MHz, which, when compared to the dispersion curves in Fig. 1(b), is near where the S0 and A0 modes begin to converge to the Rayleigh wave speed.

3.2. Strict definition of a quasi-Rayleigh wave

Although we know from Section 3.1 that the current Partial Wave Method calculation technique will not give accurate representations of the Lamb waves at a phase velocity equal to the Rayleigh wave's wave speed, we can still use the conceptual understanding of the method as a way to analyze the high frequency-thickness regime of a quasi-Rayleigh wave.

The quasi-Rayleigh wave is a superposition of the A0 and S0 Lamb wave modes. From the perspective of the Partial Wave Method, this would be the superposition of eight partial waves; four for each Lamb wave mode. For each mode there are two partial waves defined on the top surface of the plate and two partial waves defined on the bottom surface of the plate. The eigenvector representations of these partial waves are only dependent on the bulk wave speeds and the phase velocity. That being said, since the phase velocity of the S0 and A0 modes converge to the Rayleigh wave speed at high frequency-thickness, the partial waves of each mode approximate each other. Next, we consider that the S0 mode's bottom partial waves will be 180° out-of-phase with the partial waves on the top of the plate. Upon the superposition of the A0 and S0 modes, the partial waves on the bottom of the plate cancel and the partial waves on the top of the plate add. Given that the partial wave composition of each mode is essentially the same in this scenario, then the addition of partial waves at the top of the plate is a doubling of the other pairs of partial waves. That is, the scenario at high frequency-thickness can be represented in terms of two partial waves just like the Rayleigh wave. A good way to determine when this scenario can occur is to calculate the Rayleigh wave solution in a half-space and calculate the stress at a depth equal to the thickness of a plate. If the stress is such that it would meet the traction-free BC if a free surface were to be there, then it would be likely that a quasi-Rayleigh wave of this type could propagate; i.e., σ_{zz} and σ_{xz} at $z = H$ should equal zero. We start this analysis by defining the generalized Hooke's Law, the strain-displacement relationship, and the Lamé parameters in terms of the density, the longitudinal wave speed and the shear wave speed,

$$\sigma_{ij} = \lambda\, \epsilon_{kk}\, \delta_{ij} + 2\mu\, \epsilon_{ij}, \tag{6}$$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{7}$$

$$\lambda = \rho(c_L^2 - 2c_S^2), \tag{8}$$

$$\mu = \rho c_S^2. \tag{9}$$

Next, the partial-wave solution for a Rayleigh wave is constructed,

$$u_i(z,\omega) = B_L \hat{u}_i^{(-L)} e^{iz\omega\sqrt{c_L^{-2}-c_R^{-2}}} + B_S \hat{u}_i^{(-S)} e^{iz\omega\sqrt{c_S^{-2}-c_R^{-2}}}. \tag{10}$$

The partial wave eigenvectors are normalized to follow the procedure defined in [7],

$$\hat{u}_i^{(-L)} = \frac{1}{M_L}\begin{bmatrix} -\frac{c_R^{-1}}{\sqrt{c_L^{-2}-c_R^{-2}}} \\ 0 \\ 1 \end{bmatrix}, \tag{11}$$

$$\hat{u}_i^{(-S)} = \frac{1}{M_S}\begin{bmatrix} \frac{\sqrt{c_S^{-2}-c_R^{-2}}}{c_R^{-1}} \\ 0 \\ 1 \end{bmatrix}, \tag{12}$$

where M_L and M_S are the magnitude of the longitudinal wave's eigenvector and the shear wave's eigenvector, respectively. These magnitudes can be represented as,

$$M_L = \frac{\sqrt{2c_L^2 - c_R^2}}{\sqrt{c_L^2 - c_R^2}}, \tag{13}$$

$$M_S = \frac{\sqrt{2c_S^2 - c_R^2}}{c_S}. \tag{14}$$

When Eqs. (11) and (12) are substituted into Eq. (10), and then into a combination of Eqs. (6)–(9), the partial wave amplitude of the shear partial wave, B_S, can be calculated using the traction-free BC at $z = 0$,

$$B_S = -\frac{1}{2} B_L \frac{M_S}{M_L} \frac{c_L}{c_S} \frac{(2c_S^2 - c_R^2)}{\sqrt{c_L^2 - c_R^2}\sqrt{c_S^2 - c_R^2}}. \tag{15}$$

Using this expression for B_S, we can rewrite Eq. (10) as,

$$u_i(z,\omega) = \frac{B_L}{M_L}\left(\begin{bmatrix} -\frac{c_R^{-1}}{\sqrt{c_L^{-2}-c_R^{-2}}} \\ 0 \\ 1 \end{bmatrix} e^{iz\omega\sqrt{c_L^{-2}-c_R^{-2}}} - \frac{c_L(2c_S^2 - c_R^2)}{2c_S\sqrt{c_L^2 - c_R^2}\sqrt{c_S^2 - c_R^2}} \begin{bmatrix} \frac{\sqrt{c_S^{-2}-c_R^{-2}}}{c_R^{-1}} \\ 0 \\ 1 \end{bmatrix} e^{iz\omega\sqrt{c_S^{-2}-c_R^{-2}}} \right). \tag{16}$$

Now the goal is to calculate $\sigma_{xz}(z = H)/\sigma_{xx}(z = 0)$ by using Eq. (16). σ_{xz} was chosen instead of σ_{zz}, since it is known that σ_{xz} evanesces into the thickness slower than σ_{zz} [2]. The normalization procedure is the same as in [2] and is used since the maximum stress is σ_{xx} at $z = 0$. When calculated, it was found that,

$$\sigma_{xx}(z=0) = -\frac{iB_L\omega\rho(4c_R^2c_S^4 - 4c_L^2c_R^2c_S^2)}{2M_L c_S^2 c_R c_L\sqrt{c_R^2 - c_L^2}}, \tag{17}$$

$$\frac{\sigma_{xz}(z=H)}{\sigma_{xx}(z=0)} = \frac{c_S(4{c_L}^2{c_S}^4 - 4{c_L}^2{c_R}^2{c_S}^2 + {c_L}^2{c_R}^4)}{\sqrt{{c_R}^2-{c_S}^2}\,(4c_R^2c_S^4 - 4c_L^2c_R^2c_S^2)} e^{\frac{i\sqrt{{c_R}^2-{c_S}^2}\,\omega H}{c_R c_S}} + \frac{4c_L\sqrt{{c_R}^2-{c_L}^2}\,{c_S}^4}{4c_R^2c_S^4 - 4c_L^2c_R^2c_S^2} e^{\frac{i\sqrt{{c_R}^2-{c_L}^2}\,\omega H}{c_L c_R}}. \tag{18}$$

Since the stress as a function of depth only approaches zero as $z \to \infty$, we arbitrarily define the strict region as,

$$\left|\frac{\sigma_{xz}(z=H)}{\sigma_{xx}(z=0)}\right| \leq 0.001. \tag{19}$$

Due to the normalization, another way of expressing this condition is to write that the strict definition of the quasi-Rayleigh wave begins when the traction at $z = H$ is 0.1% of the maximum stress of the Rayleigh wave (i.e., the maximum stress is σ_{xx} at $z = 0$). Next, we solve Eq. (19) for the frequency-thickness product, but an analytical solution would be difficult, so we note that the first term in Eq. (16) should evanesce slower than the second term since the difference between the shear wave speed and the Rayleigh wave speed is smaller than the longitudinal wave speed and the Rayleigh wave speed for most isotropic solids. Using this, we assume the first term will be representative of the function as $z \to 0$ and ignore the second term. This results in,

$$f_c^{strict}H \approx \frac{-ic_Rc_S}{2\pi\sqrt{c_R{}^2-c_S{}^2}}\ln\left(\frac{0.001i\sqrt{c_R{}^2-c_S{}^2}\,(4c_R^2c_S^4-4c_L^2c_R^2c_S^2)}{c_S\,(4c_L{}^2c_S{}^4-4c_L{}^2c_R{}^2c_S{}^2+c_L{}^2c_R{}^4)}\right) \tag{20}$$

Viktorov's condition (Eq. (2) can also be rearranged into an expression for a cut-off frequency by using the relationship $c_R = \lambda_R f$,

$$f_c^{Victorov}H \geq 2c_R. \tag{21}$$

For a 1-mm aluminum plate, Eq. (18) is plotted with respect to frequency in Fig. 4. Also shown in Fig. 4 are the cut-off frequencies for Viktorov's condition and the strict definition, which are $f_c^{Viktorov} = 5.79$ MHz and $f_c^{strict} = 9.28$ MHz, respectively.

An example of the superposition of the A0 and S0 wave-structures at 9.28 MHz is shown in Fig. 5 and the superposition can be seen to closely resemble a Rayleigh wave. The wave-structures of the S0 and A0 modes used in Fig. 5 were calculated by the SAFE method. It's possible to infer from Fig. 5 that with an increase in the phase difference of π, the Rayleigh-wave-like wave-structure would flip surfaces. The normalized stress is zero (in Fig. 4) at very low frequencies because the traction-free boundary condition is satisfied at $z = 0$ and the stress distribution through the thickness is essentially planar for very low frequencies (i.e., for low frequencies, the stress at $z = H$ is approximately equal to stress at $z = 0$, which is zero due to the traction-free BC).

As we will see in the following subsection, this strict definition for a quasi-Rayleigh wave is not the only way a quasi-Rayleigh wave can propagate.

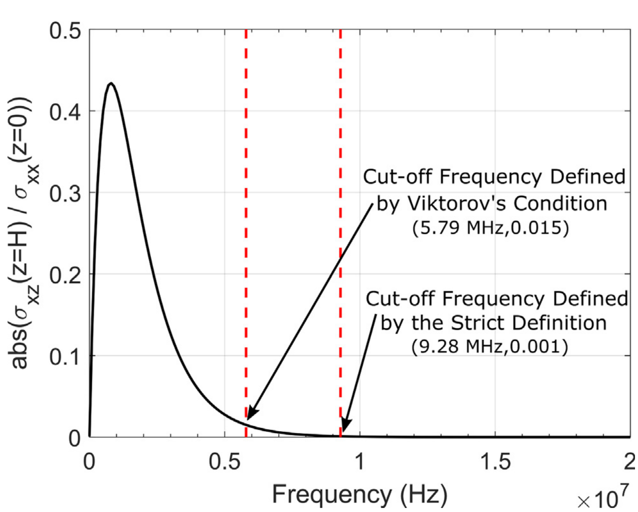

Fig. 4. Normalized σ_{xz} stress at $z = H$ for a Rayleigh wave in a half-space. The stress was normalized with respect to the maximum σ_{xx} which occurs at $z = 0$. Note that the stress never goes exactly to zero for finite z, as a result of the function's exponential behavior.

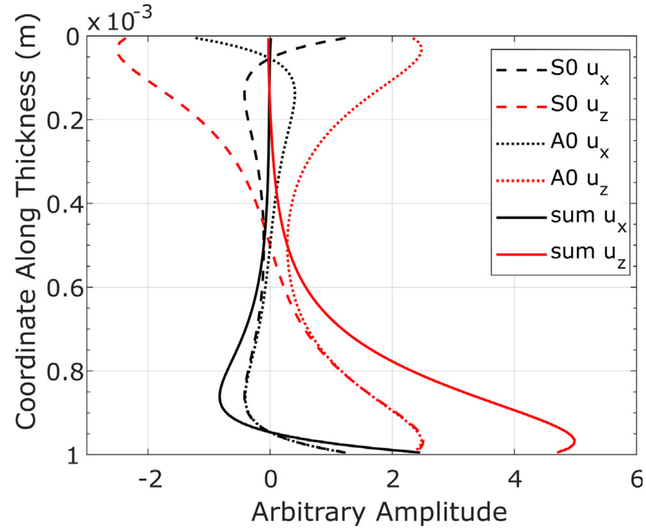

Fig. 5. The wave-structures of the S0 and A0 Lamb waves and their superposition at 9.28 MHz with a phase difference of $3\pi/2$ rad.

3.3. Lenient definition of a Quasi-Rayleigh wave

Section 3.2 specifies a strict definition for a quasi-Rayleigh wave, but many of the experimental measurements [3,4,9] and Viktorov's condition, observe and predict quasi-Rayleigh waves at lower frequencies. In fact, some experiments [3,9] are actually performed below the cut-off frequency specified by Viktorov's condition for their respective waveguides. With the results of Sections 3.1 and 3.2 in mind, we now seek to use the Partial Wave Method, where accurate, to expand the definition of a quasi-Rayleigh wave while keeping in mind the two conditions defined at the beginning of the section.

The superposition of the S0 and A0 Lamb wave modes is difficult to analyze because they can be scaled or normalized in various ways. Depending on how this is done, it can be difficult to objectively determine whether a wave-structure from a superposition "looks" like a Rayleigh wave let alone whether it can be classified as a quasi-Rayleigh wave. In this subsection we will attempt to make the process of classifying a quasi-Rayleigh wave more quantifiable.

To do this, similar to the strict definition, the concept of a superposition of eight partial waves is used. However, we no longer require the partial waves to cancel completely. That is, the A0 and S0 modes do not need to have approximately the same phase velocity, or even group velocity. The wave-structure, as per condition (1), need only resemble a Rayleigh wave and as per condition (2), exhibit a beat phenomenon.

To solve the problem of scaling and normalization, the ratio of partial wave amplitudes, specifically B_L/B_{SV}, where B_L and B_{SV} are the coefficients of the longitudinal and shear vertical partial waves respectively (see [7]), will be used to compare the structure of the A0 and S0 modes. This ratio will be used to compare the composition of the two modes with the similar goal of having constructive interference on one surface and destructive interference on the other. It should be noted that this ratio is not indicative of the 180° phase difference between the top and bottom partial waves of the S0 modes. The strict definition would be considered the ideal case for a ratio comparison, in which the A0 and S0 partial wave amplitude ratios are equivalent, which when considered with the S0 mode's characteristic phase difference between surfaces would mean that an ideal quasi-Rayleigh wave would form. For this lenient definition we do not require that the ratio be equivalent, but they must be of similar type such that when superposed they add up when in-phase and cancel when 180° out-of-phase. The amplitude ratio for the S0 and A0 modes with respect to frequency is plotted in Fig. 6.

Since most of the features of Fig. 6 are between 0 and 5 MHz (see Fig. 3), the plot shown is an accurate representation of the partial wave

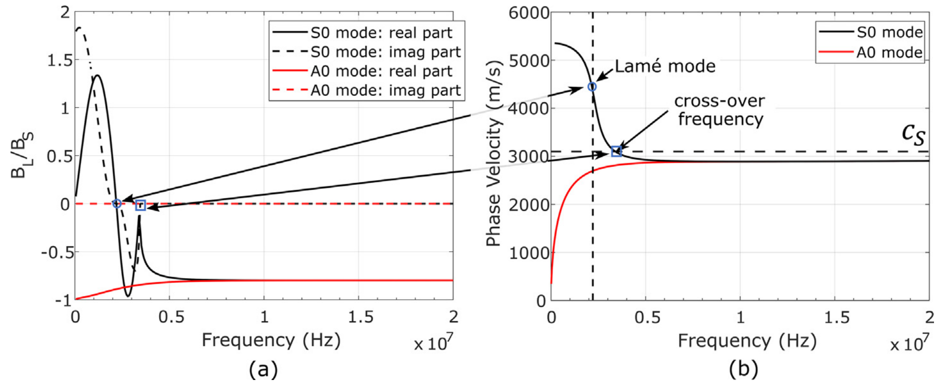

Fig. 6. (a) The amplitude ratio (i.e., B_L/B_S) between the longitudinal and shear-vertical partial waves for the S0 and A0 modes from 0 to 20 MHz. (b) The S0 and A0 dispersion curves are provided for easy comparison. The cross-over frequency is the frequency at which the S0 dispersion curve (going from high frequency to low frequency) crosses from region 1 ($c_p < c_S$) to region 2 ($c_S < c_p < c_L$). See Hakoda and Lissenden [7] for more information on regions 1 and 2.

composition. As shown in Fig. 6, the amplitude ratio for the A0 mode remains real for the entire frequency range, which is in part due to the A0 mode remaining in region 1 ($c_p < c_S$). That is, the composition of the A0 mode is made up of only surface partial waves, which is ideal for potential Rayleigh wave propagation. The amplitude ratio of the S0 mode on the other hand clearly changes from real- to complex-valued after crossing the vertical dotted line, which is the frequency at which the S0 dispersion curve crosses over from region 1 to region 2 ($c_S < c_p < c_L$) of the dispersion curve space (see Hakoda and Lissenden [7] for a detailed description of regions 1–3). This frequency will be referred to as the crossover frequency for brevity.

For frequencies greater than the crossover frequency, the S0 mode is made up of surface partial waves, since it is within region 1. The S0 mode also has an amplitude ratio that is close to the A0 amplitude ratio, but the amplitude ratio approaches zero as the frequency approaches the crossover frequency. At frequencies below the crossover frequency, the amplitude ratio becomes zero at a frequency which corresponds with the Lamé wave mode, as described by Auld [10], and marks a transition from a negative amplitude ratio to a positive amplitude ratio. The phase velocity of the Lamé wave mode can be calculated by knowing that it is a Lamb wave composed of only shear-vertical partial waves obliquely reflecting from the plate surfaces at 45° angles,

$$c_x = c_S\sqrt{2}. \tag{22}$$

The frequency-thickness product at which Lamé wave modes propagate can be calculated to be $fH = \frac{nc_S}{\sqrt{2}}$ where $n = 0, 1, 2, \cdots$ according to Graff [11]. For the lower bound of the lenient definition, the Lamé wave mode where $n = 1$ will be used which results in the cut-off frequency of the lenient definition being,

$$f_c^{lenient} H = \frac{c_S}{\sqrt{2}} \tag{23}$$

For the 1-mm aluminum plate example, this would result in $f_c^{lenient} = 2.19$ MHz.

For frequencies below the crossover frequency, the amplitude ratio is complex-valued and the partial wave is no longer composed of only surface waves. The longitudinal partial wave remains as a surface wave, but the shear-vertical partial wave can now be represented as an obliquely travelling bulk wave. Through the thickness, instead of an exponentially decaying or increasing function that is characteristic of a surface wave, the shear-vertical partial wave appears as a cosine and sine wave. This is, of course, not characteristic of a Rayleigh wave, let alone a quasi-Rayleigh wave. However, when paired with the longitudinal surface partial wave it can approximate a Rayleigh wave when the amplitude ratio is negative-valued. This is difficult to comprehensively quantify since the partial wave's complex-valued eigenvectors would complicate the superposition of partial waves, but example calculations using wave-structures from both the Partial Wave Method and the SAFE method support this trend. Based on this criterion, the frequency for the Lamé wave mode (as defined by Auld [10]) can be used as a cut-off frequency for the more lenient definition of the quasi-Rayleigh wave. This type of quasi-Rayleigh wave has a wave-structure that only resembles a Rayleigh wave, the wave energy is noticeably larger on one surface when compared to the other, and it exhibits the beat phenomenon. An example of what the beat phenomenon and wave-structure look like is shown in Fig. 7, which uses wave-structures calculated using the SAFE method at a frequency of 2.67 MHz. At frequencies below the Lamé wave mode frequency, the superposition of the A0 and S0 mode loses all of these characteristics. An example of what the superposition looks like at a frequency of

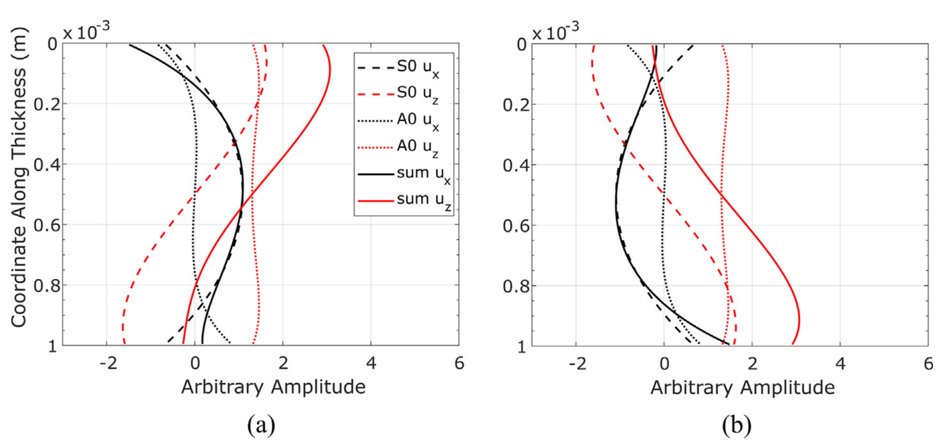

Fig. 7. The wave-structures of the S0 and A0 Lamb waves and their superposition at 2.67 MHz with a phase difference of (a) $\pi/2$ rad and (b) $3\pi/2$ rad.

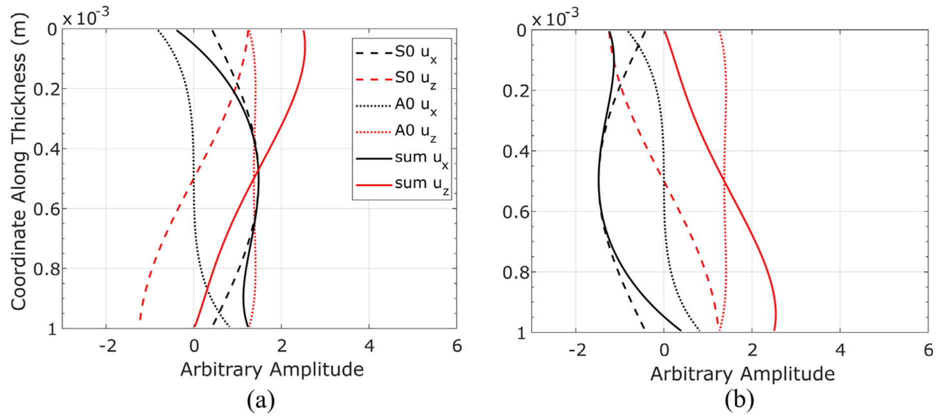

Fig. 8. The wave-structures of the S0 and A0 Lamb waves and their superposition at 2 MHz with a phase difference of (a) $\pi/2$ rad and (b) $3\pi/2$ rad.

2 MHz, which is just below the Lamé wave mode's frequency, is shown in Fig. 8. That is, the x-component of the displacement, u, is no longer small at the surface opposite the Rayleigh-like wave, which means the wave-structure is no longer Rayleigh-like and the wave does not have the characteristic beat phenomenon.

To look at the results from a different perspective, the S0 mode's amplitude ratio for frequencies less than the crossover frequency are plotted in the complex plane in Fig. 9. Since the amplitude ratio of the A0 modes surface partial waves are real and negative, they would lie along the left-hand side of the horizontal axis. Over this frequency range, the minimum amplitude ratio would be -1 for the A0 mode. Based on the criterion we set, as long as the amplitude ratio of the S0 mode is within quadrant 3 then a quasi-Rayleigh wave can propagate. Quasi-Rayleigh waves between the crossover frequency and the cut-off frequency of the Strict Definition frequency range can also be included in the Lenient Definition, since the final quasi-Rayleigh wave has similar characteristics. However, using the Partial Wave Method for the calculation of partial wave amplitude composition is not recommended in this region because of the aforementioned lopsidedness and inability to meet the fourth traction-free boundary condition.

4. Conclusion

In this paper, we analyzed the characteristics of quasi-Rayleigh waves as described in the literature and then used the Partial Wave Method to classify frequency regions in which various types of quasi-Rayleigh waves exist. A summary of these frequency regions with

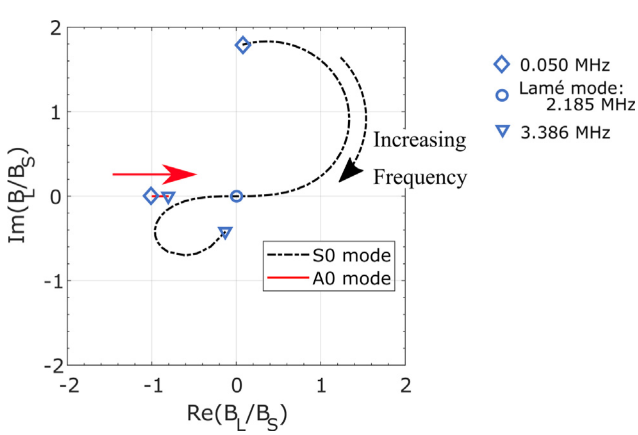

Fig. 9. The real and imaginary parts of the A0 and S0 mode's amplitude ratios for frequencies less than the crossover frequency. The arrows which matching the line pattern and color of the S0 and A0 mode plots designate the direction of increasing frequency. For example, The A0 mode's amplitude ratio goes from -1 to around -0.8 as the frequency increases.

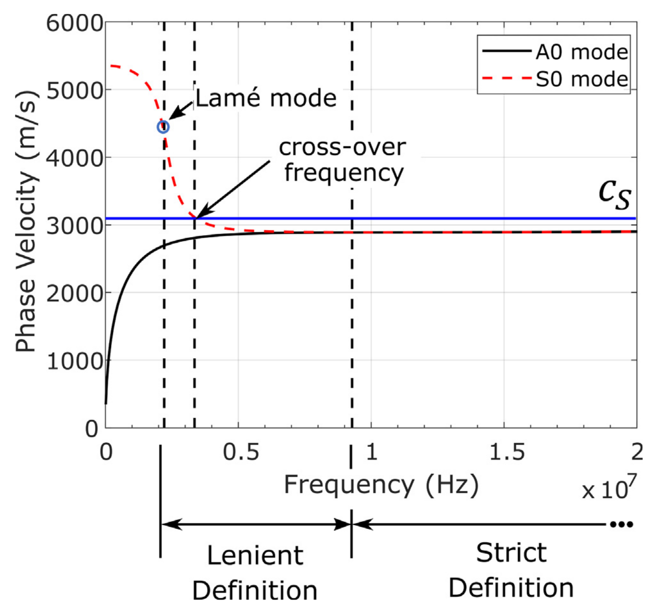

Fig. 10. A summary figure to show the frequency regions of the strict and lenient definition of a quasi-Rayleigh wave with respect to the dispersion curves from a 1-mm-thick aluminum plate waveguide.

respect to the dispersion curves of a 1-mm aluminum plate waveguide are shown in Fig. 10. Specifically, the lenient definition is formally from the frequency at which the S0 dispersion curve is at the Lamé mode's phase velocity as defined by Auld [10], to the frequency at which the S0 dispersion curve crosses from region 1 to region 2 of the dispersion curve space. For a 1-mm thick aluminum plate this would be from 2.1 MHz to 3.4 MHz. However, the definition can be extended to include frequencies up to the cut-off frequency of the strict definition. The strict definition, as defined in this paper, begins when the normalized traction of the Rayleigh wave at $z = H$ is 0.001, which for a 1-mm aluminum plate is 9.28 MHz. The strict definition's frequency range includes all frequencies larger than this cut-off frequency.

Quasi-Rayleigh waves in the frequency range of the strict definition closely resemble the wave-structure of a Rayleigh wave and exhibit the beat phenomenon. However, quasi-Rayleigh waves in the frequency range of the lenient definition have wave-structures that only vaguely resemble a Rayleigh wave, but still exhibit the beat phenomenon. Most indicative of this is that the displacement wave-structure does not go to zero on the surface opposite the Rayleigh-wave-like wave-structure. Also of note, the cut-off frequency derived from Viktorov's condition and the experimental observations in the literature [3,4,9] are all within the frequency range of the lenient definition. The quasi-Rayleigh wave excited in Masserey and Fromme [4] is the closest of the three papers to the strict definition.

References

[1] I. Viktorov, R. Grigoryan, Quasi-Rayleigh waves in an elastic layer, Akusticheskij Zhurnal 5 (3) (1959) 171–175.
[2] I. Viktorov, Rayleigh and Lamb Waves: Physical Theory and Applications, Springer Science + Business Media, New York, 1967.
[3] H. Michal, S. Ladislav, M. Milos, H. Branislav, Ultrasonic defect detection fo structural plates using quasi-Rayleigh waves, J. Mech. Eng. 67 (2) (2017) 37–50.
[4] B. Masserey, P. Fromme, On the reflection of coupled Rayleigh-like waves at surface defects in plates, J. Acoust. Soc. Am. 123 (88) (2008) 88–98.
[5] L. Brekhovskikh, V. Goncharov, Mechanics of Continua and Wave Dynamics, Springer-Verlag, Berlin, 1985.
[6] J. Rose, Ultrasonic Waves in Solid Media, Cambridge University Press, New York, 1999.
[7] C. Hakoda, C. Lissenden, Using the Partial Wave Method for wave-structure calculation and the conceptual interpretation of elastodynamic guided waves, Appl. Sci.: Acoust. 8 (6) (2018).
[8] COMSOL Inc., COMSOL Multiphysics: Reference Manual, Stockholm: COMSOL Inc., 2015.
[9] B. Ti, W. O'Brien, J. Harris, Measurements of coupled Rayleigh wave propagation in an elastic plate, J. Acoust. Soc. Am. 102 (3) (1997) 1528.
[10] B. Auld, Acoustic Fields and Waves in Solids, Malabar, Robert E. Krieger Publishing Company, Florida, 1990.
[11] K.F. Graff, Wave Motion in Elastic Solids, Dover Publications, INC., New York, 1991.