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A Novel Decision-Making Method Based on Probabilistic Linguistic Information

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Abstract

The Maclaurin symmetric mean (MSM) operator has the characteristic of capturing the interrelationship among multi-input arguments, the probabilistic linguistic terms set (PLTS) can reflect the different degrees of importance or weights of all possible evaluation values, and the improved operational laws of probabilistic linguistic information (PLI) can not only avoid the operational values out of bounds for the linguistic terms set (LTS) but also keep the probability information complete after operations; hence, it is very meaningful to extend the MSM operator to PLTS based on the operational laws. To fully take advantage of the MSM operator and the improved operational laws of PLI, the MSM operator is extended to PLI. At the same time, two new aggregated operators are proposed, including the probabilistic linguistic MSM (PLMSM) operator and the weighted probabilistic linguistic MSM (WPLMSM) operator. Simultaneously, the properties and the special cases of these operators are discussed. Further, based on the proposed WPLMSM operator, a novel approach for multiple attribute decision-making (MADM) problems with PLI is proposed. With a given numerical example, the feasibility of the proposed method is proven, and a comparison with the existing methods can show the advantages of the new method in this paper. The developed method adopts the new operational rules with the accurate operations, and it can overcome some existing weaknesses and capture the interrelationship among the multi-input PLTSs, which easily express the qualitative information given by the decision-makers' cognition.

Keywords Probabilistic linguistic · Maclaurin symmetric mean · Multiple attribute decision-making

Introduction

Multiple attribute decision-making (MADM) problems exist widely in real life, such as venture capital problems [46], hospital resource allocation problems [4], and green supply chain selection problems [44]. To solve these problems, people used to express the evaluation information by the quantitative value. However, along with the increase in the uncertainty and complexity of the decision environment as well as the ambiguity of human cognition, it is difficult for people to provide a quantitative value; sometimes, it may be more convenient to give a qualitative description. For example, when the risk of an investment object is evaluated, decision-makers (DMs) are more likely to use "high," "medium," "low," and other similar linguistic terms (LTs) to express their assessment results [7, 34], i.e., LTs are more consistent with people's cognitive behaviors. Furthermore, due to the lack of knowledge of DMs and the complexity of decision-making problems and so on, DMs are usually unable to use precise numbers to describe their preferences. In such a situation, LTs are a useful and practical form for expressing the cognition and judgments of DMs. Therefore, decision-making based on LTs has become an important research direction in the field of decision analysis. Motivated by this idea, Zadeh [39–41] first proposed the linguistic variable (LV) which provided the foundation of later studies on the linguistic MADM based on a linguistic terms set (LTS). Furthermore, LVs are extended to some new types for different fuzzy information, such as intuitionistic linguistic sets which were produced by combining intuitionistic fuzzy sets (IFSs) [15] with LVs, intuitionistic linguistic numbers [32], intuitionistic uncertain LV (IULV) [19], and interval-valued IULV [14].

Using LVs, DMs express their preferences or assess judgments only by one LT. In practice, however, DMs may have some hesitations on several possible LTs. To address this situation, Rodriguez et al. [29] proposed the hesitant fuzzy LTSs

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(HFLTSs). HFLTSs collect some possible LTs provided by the DMs, and all of these terms have an equal degree of importance or weight. However, in some practical cases, there are different degrees of importance among these possible LTs. In other words, the probabilities of the LTs are not equal. Thus, the HFLTSs may cause information loss, so Zhu [48] proposed the extended hesitant fuzzy sets to overcome this defect. Further, Pang et al. [26] extended the HFLTSs to a more general concept, probabilistic LTSs (PLTSs), which allow DMs to provide more than one LT with probability. Based on the advantages of PLTSs, Pang et al. [26] extended the TOPSIS method to process PLTSs; Zhang et al. [46] proposed the probabilistic linguistic preference relationship. However, the existing operational laws of LTs in PLTSs have some shortcomings which may exceed the bounds of LTS. To avoid these weaknesses, Gou et al. [9, 10] proposed two equivalent transformation functions, g and g^{-1}, and then defined some novel operational laws of PLTSs, which can not only avoid exceeding the bounds of operational results but also retain the probabilistic information after operations. Based on the novel operational laws [9, 10], Bai et al. [4] developed the possibility degree formula for PLTSs and applied it to solve the hospital resource allocation problem. Zhang and Xing [44] combined the PLTSs with the VIKOR method to solve the green supply chain selection problems.

In recent years, some extended aggregation operators have been proposed [13–25, 27, 31, 42, 43] for the different functions. For example, Rong et al. [30] proposed the Hamacher operators for IFNs, which can provide more general operations. Baležentis [2] proposed some power aggregation operators for intuitionistic trapezoidal fuzzy numbers (ITFNs), which consider the attribute values to provide support for each other. Liu et al. [17] proposed the prioritized ordered weighted averaging (POWA) operator for ITFNs, which can consider the precedence relationship among the attributes. Xu et al. [37] proposed the Bonferroni mean (BM) operators for intuitionistic fuzzy numbers (IFNs); He et al. [11], and He and He [12] proposed the interaction BMs operators for IFNs; Yu et al. [38] proposed the Heronian mean (HM) operators for interval-valued IFNs (IVIFNs); these operators can consider the interrelationships between two aggregating parameters. However, in real decision problems, there exist interrelationships among multi-input parameters; clearly, BM and HM operators cannot achieve this function. Thus, the MSM operators were proposed by Maclaurin [24], and they can consider the interrelationships among any number of multi-input parameters. In the past few years, MSM has received increasing attention; many important results are achieved in theory [1, 3, 5], and some extensions are developed. Zhang [45] proposed the S-geometric convexity of a function involving Maclaurin's elementary symmetric mean. Gao [8] investigated some inequalities involved the MSM operator and provided a strictly monotonic proof of the Maclaurin inequality. Zhang et al. [47] developed a general family of weighted elementary symmetric means and discussed some of their mathematical properties. Detemple and Robertson [6] proposed the generalized MSM operators in which both the individual importance and their interactions were considered. Qin and Liu [28] developed some MSM operators for IFNs and applied them to solve MADM problems. Because the MSM can consider the interrelationships among the multi-input arguments, it can provide a more flexible way for information fusion, and it is also more adequate for solving the MADM problems in which the attributes are dependent. Moreover, for a given collection of arguments, the MSM decreases monotonically with its parameter value, which can reflect DMs' risk preferences in practical situations. In addition, the MSM has been applied to solve practical MADM problems, such as company investment problems [33].

As discussed above, we can see that PLTSs allow DMs to express their preferences on some LTs with different probabilities, and the MSM operator can consider the interrelationship among the multi-input arguments and also reflect the risk preferences of DMs in practical situations. However, the traditional MSM operator only considers the case where the arguments take the form of crisp numbers, and it cannot process the PLI. Therefore, it is meaningful and necessary to extend the MSM to aggregate PLI. Motivated by this idea, the goal and contributions of this paper are (1) to extend MSM operator to PLI and propose some probabilistic linguistic MSM operators; (2) to explore some desirable properties and special cases of these proposed operators; (3) to develop a MADM method based on the proposed aggregation operators; and (4) to show the feasibility and advantages of the proposed methods.

To achieve the above goal, the remainder of this paper is set as follows. "Preliminaries" provides some basic concepts of PLI and the MSM operator. In "PLMSM Operators," some probabilistic linguistic MSM (PLMSM) operators are proposed, and their properties and some special cases are discussed. In "A Decision-Making Method Based on the WPLMSM Operator," a MADM method is developed based on the proposed operators with PLI. In "Numerical Example," an example is given to illustrate the effectiveness of the proposed method. "Conclusion" provides the conclusions and the direction of future studies.

Preliminaries

First, in this part we provide some preliminaries about this study.

Additive Linguistic Evaluation Scales

An additive linguistic evaluation scale (also called LTS) is one of the most commonly used linguistic evaluation scales, and it is usually given in the following form: $S = \{s_\alpha | \alpha = 0, 1, ..., \tau\}$, where s_α represents an LV, and τ is a positive integer.

For example, when two investment projects A and B are evaluated, the DMs give their evaluation values based on the LTS $S = \{s_0 = very\ bad, s_1 = bad, s_2 = medium, s_3 = good, s_4 = very\ good\}$. If a DM prefers A to B, he/she will use "positive" linguistic labels, i.e., "good" or "very good," to express his/her preference. Different "positive" linguistic labels reflect different degrees of preference of the DMs.

To avoid information distortion, the discrete LTS is extended to a continuous one. For LTS $S_1 = \{s_\alpha | \alpha = 0, 1, ..., \tau\}$, Xu [36] introduced two basic operational laws based on its continuous form $\overline{S}_1 = \{s_\alpha | \alpha \in [0, \tau]\}$: (1) $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$; (2) $\lambda s_\alpha = s_{\lambda\alpha}$, where $s_\alpha, s_\beta \in \overline{S}_1$ and $\lambda \in [0, 1]$. However, in the process of operations, if $s_1 = bad$ and $s_2 = medium$, then $s_1 \oplus s_2 = s_3$, i.e., the additive operational result of the linguistic labels "bad" and "medium" is "good." Obviously, this result does not conform to human intuition. To overcome this shortcoming, Xu [35] extended the discrete LTS to subscript-symmetric LTS, $S_2 = \{s_\alpha | \alpha = -\tau, ... -1, 0, 1, ..., \tau\}$, where the mid-linguistic label s_0 represents the meaning of "indifference," and the rest are placed symmetrically around it. Here, we still use the example of evaluating two investment projects A and B; based on a subscript symmetric LTS $S_2 = \{s_\alpha | \alpha = -2, -1, 0, 1, 2\}$, the DMs give their evaluation values among the LTS $\{s_{-2} = very\ bad, s_{-1} = bad, s_0 = medium, s_1 = good, s_2 = very\ good\}$. In this case, $s_{-1} + s_0 = s_{-1}$, i.e., the additive operational result of the linguistic labels "bad" and "medium" is "bad." Clearly, this is more reasonable than using LTS S_1. However, when we calculate $s_{-2} + s_{-1} = s_{-3}$, this result is still out of bounds, and it is unreasonable.

PLTS

Definition 1 [26] Let $S_1 = \{s_\alpha | \alpha = 0, 1, ..., \tau\}$ be an LTS, and PLTS is defined as:

$$LS(p) = \left\{ LS^{(k)}\left(p^{(k)}\right) | LS^{(k)} \in S_1, p^{(k)} \geq 0, k = 1, 2, ..., \#LS(p), \sum_{k=1}^{\#LS(p)} p^{(k)} \leq 1 \right\} \quad (1)$$

where $LS^{(k)}(p^{(k)})$ represents the LT $LS^{(k)}$ with the probability $p^{(k)}$, and $\#LS(p)$ is the number of all different LTs in $LS(p)$.

Clearly, Pang et al. [26] defined the PLTS on the basis of the LTS S_1. In this paper, we use the PLTSs based on the LTS S_2, and the PLTSs can be expressed as:

$$LS(p) = \left\{ LS^{(k)}\left(p^{(k)}\right) | LS^{(k)} \in S_2, p^{(k)} \geq 0, k = 1, 2, ..., \#LS(p), \sum_{k=1}^{\#LS(p)} p^{(k)} \leq 1 \right\} \quad (2)$$

Normalization of PLTS

Definition 2 [26] Given a PLTS $LS(p)$ with $\sum_{k=1}^{\#LS(p)} p^{(k)} < 1$, the associated PLTS $\dot{LS}(p) = \left\{ LS^{(k)}\left(\dot{p}^{(k)}\right) | k = 1, 2, ..., \#LS(p) \right\}$ is called a normalized PLTS, where $\dot{p}^{(k)} = p^{(k)} / \sum_{k=1}^{\#LS(p)} p^{(k)}$ for all $k = 1, 2, ..., \#LS(p)$. Clearly, in PLTS $\dot{LS}(p)$, there is $\sum_{k=1}^{\#LS(p)} \dot{p}^{(k)} = 1$.

Definition 3 [26] For any two PLTSs $LS_1(p) = \{LS_1{}^{(k)}(p_1{}^{(k)}) | k = 1, 2, ..., \#LS_1(p)\}$ and $LS_2(p) = \{LS_2{}^{(k)}(p_2{}^{(k)}) | k = 1, 2, ..., \#LS_2(p)\}$, suppose $\#LS_1(p)$ and $\#LS_2(p)$ are the numbers of LTs in $LS_1(p)$ and $LS_2(p)$, respectively. If $\#LS_1(p) > \#LS_2(p)$, then $\#LS_1(p) - \#LS_2(p)$ LTs will be added to $LS_2(p)$ such that their numbers of LTs are equal. For detailed information, please refer to reference [26].

By Definitions 2 and 3, the normalized PLTS (NPLTS) is obtained, which is denoted as $LS^N(p) = \{LS^{N(k)}(p^{N(k)}) | k = 1, 2, ..., \#LS(p)\}$, where $p^{N(k)} = p^{(k)} / \sum_{k=1}^{\#LS(p)} p^{(k)}$ for all $k = 1, 2, ..., \#LS(p)$.

Comparison of PLTSs

To compare the PLTSs, Pang et al. [26] presented the concept of the score for PLTSs:

Definition 4 [26] Suppose $LS(p) = \{LS^{(k)}(p^{(k)}) | k = 1, 2, ..., \#LS(p)\}$ is a PLTS, and $r^{(k)}$ is the subscript of the LT $LS^{(k)}$. Then, the score function $E(LS(p))$ of $LS(p)$ is given by

$$E(LS(p)) = s_{\overline{\alpha}} \quad (3)$$

where $\overline{\alpha} = \sum_{k=1}^{\#LS(p)} \left(r^{(k)} p^{(k)}\right) / \sum_{k=1}^{\#LS(p)} p^{(k)}$.

Apparently, the score function represents the average value of all LTs of a PLTS. Generally, for a given PLTS $LS(p)$, $E(LS(p))$ is an extended LT.

Definition 5 [26] For given any two PLTSs $LS_1(p)$ and $LS_2(p)$, if $E(LS_1(p)) > E(LS_2(p))$, then the PLTS $LS_1(p)$ is greater than $LS_2(p)$.

However, when $E(LS_1(p)) = E(LS_2(p))$, we cannot compare the PLTSs $LS_1(p)$ and $LS_2(p)$. Further, to process this situation, we give the following definition.

Definition 6 [26] For a given PLTS $LS(p) = \{LS^{(k)}(p^{(k)}) | k = 1, 2, ..., \#LS(p)\}$, suppose $r^{(k)}$ is the subscript of LT $LS^{(k)}$, and

$E(LS(p)) = s_{\overline{\alpha}}$, where $\overline{\alpha} = \sum_{k=1}^{\#LS(p)} \left(r^{(k)} p^{(k)}\right) / \sum_{k=1}^{\#LS(p)} p^{(k)}$. Then, the degree of deviation of $LS(p)$ is:

$$\sigma(LS(p)) = \left(\sum_{k=1}^{\#LS(p)} \left(p^{(k)} \left(r^{(k)} - \overline{\alpha} \right) \right)^2 \right)^{\frac{1}{2}} / \sum_{k=1}^{\#LS(p)} p^{(k)} \quad (4)$$

For any two PLTSs $LS_1(p)$ and $LS_2(p)$ with $E(LS_1(p)) = E(LS_2(p))$, if $\overline{\sigma}(LS_1(p)) > \overline{\sigma}(LS_2(p))$, then $LS_1(p) < LS_2(p)$; if $\overline{\sigma}(LS_1(p)) = \overline{\sigma}(LS_2(p))$, then $LS_1(p)$ is indifferent to $LS_2(p)$, denoted by $LS_1(p) \approx LS_2(p)$.

Definition 7 [26] For given any two PLTSs $LS_1(p)$ and $LS_2(p)$, suppose $\tilde{L}S_1(p)$ and $\tilde{L}S_2(p)$ are the corresponding normalized PLTSs, respectively. Then

(1) *If* $E\left(\tilde{L}S_1(p)\right) > E\left(\tilde{L}S_2(p)\right)$, *then* $LS_1(p) > LS_2(p)$
(2) *if* $E\left(\tilde{L}S_1(p)\right) < E\left(\tilde{L}S_2(p)\right)$, *then* $LS_1(p) < LS_2(p)$
(3) *if* $E\left(\tilde{L}S_1(p)\right) = E\left(\tilde{L}S_2(p)\right)$, *then*

(i) *if* $\overline{\sigma}\left(\tilde{L}S_1(p)\right) > \overline{\sigma}\left(\tilde{L}S_2(p)\right)$, *then* $LS_1(p) < LS_2(p)$
(ii) *if* $\overline{\sigma}\left(\tilde{L}S_1(p)\right) = \overline{\sigma}\left(\tilde{L}S_2(p)\right)$, *then* $LS_1(p) \approx LS_2(p)$
(iii) *if* $\overline{\sigma}\left(\tilde{L}S_1(p)\right) < \overline{\sigma}\left(\tilde{L}S_2(p)\right)$, *then* $LS_1(p) > LS_2(p)$

Some Basic Operations of PLTSs

Definition 8 [26] Let $S = \{s_\alpha | \alpha = -\tau, \ldots -1, 0, 1, \ldots, \tau\}$ be an LTS, $LS(p)$, $LS_1(p)$, and $LS_2(p)$ be three PLTSs, and λ be a positive real numbers; then

$$LS_1(p) \oplus LS_2(p) = \cup_{LS_1^{(k)} \in LS_1(p), LS_2^{(k)} \in LS_2(p)} \left\{ p_1^{(k)} LS_1^{(k)} \oplus p_2^{(k)} LS_2^{(k)} \right\} \quad (5)$$

$$LS_1(p) \otimes LS_2(p) = \cup_{LS_1^{(k)} \in LS_1(p), LS_2^{(k)} \in LS_2(p)} \left\{ \left(LS_1^{(k)}\right)^{p_1^{(k)}} \oplus \left(LS_2^{(k)}\right)^{p_2^{(k)}} \right\} \quad (6)$$

$$\lambda LS(p) = \cup_{LS^{(k)} \in LS(p)} \lambda p^{(k)} LS^{(k)}, \lambda \geq 0 \quad (7)$$

$$(LS(p))^\lambda = \cup_{LS^{(k)} \in LS(p)} \left\{ \left(LS^{(k)}\right)^{\lambda p^{(k)}} \right\} \quad (8)$$

However, the above operational laws have some shortcomings in processing the LTs. To overcome these weaknesses, Gou et al. [9, 10] proposed two equivalent transformation functions g and g^{-1} and developed new operational laws, which can not only avoid the operational results exceeding the bounds of LTSs but also keep the probability information complete after operations.

Definition 9 [9, 10] Let $S = \{s_\alpha | \alpha = -\tau, \ldots -1, 0, 1, \ldots, \tau\}$ be an LTS, $LS(p)$, $LS_1(p)$, and $LS_2(p)$ be three PLTSs, and λ be a positive real numbers. $\eta^{(k)} \in g(LS(p))$, $\eta_1^{(i)} \in g(LS_1(p))$, $\eta_1^{(i)} \in g(LS_2(p))$ and $k = 1, 2, \ldots, \# LS(p)$, $i = 1, 2, \ldots, \# LS_1(p)$, $j = 1, 2, \ldots, \# LS_2(p)$ where g is the equivalent transformation function [10].

$$g : [-\tau, \tau] \rightarrow [0, 1], g(LS(p)) = \left\{ \left[\frac{r^{(k)}}{2\tau} + \frac{1}{2} \right] \left(p^{(k)} \right) \right\} = LS_\gamma(p), \gamma \in [0, 1],$$
$$g^{-1} : [0, 1] \rightarrow [-\tau, \tau], g^{-1}(LS_\gamma(p)) = \left\{ s_{(2\gamma-1)\tau} \left(p^{(k)} \right) | \gamma \in [0, 1] \right\} = LS(p).$$

$$LS_1(p) \oplus LS_2(p) = g^{-1} \left(\cup_{\eta_1^{(i)} \in g(LS_1(p)), \eta_2^{(j)} \in g(LS_2(p))} \left\{ \left(\eta_1^{(i)} + \eta_2^{(j)} - \eta_1^{(i)} \eta_2^{(j)} \right) \left(p_1^{(i)} p_2^{(j)} \right) \right\} \right) \quad (9)$$

$$LS_1(p) \otimes LS_2(p) = g^{-1} \left(\cup_{\eta_1^{(i)} \in g(LS_1(p)), \eta_2^{(j)} \in g(LS_2(p))} \left\{ \left(\eta_1^{(i)} \eta_2^{(j)} \right) \left(p_1^{(i)} p_2^{(j)} \right) \right\} \right) \quad (10)$$

$$\lambda LS(p) = g^{-1} \left(\cup_{\eta^{(k)} \in g(LS(p))} \left\{ \left(1 - \left(1 - \eta^{(k)} \right)^\lambda \right) \left(p^{(k)} \right) \right\} \right) \quad (11)$$

$$(LS(p))^\lambda = g^{-1} \left(\cup_{\eta^{(k)} \in g(LS(p))} \left\{ \left(\eta^{(k)} \right)^\lambda \left(p^{(k)} \right) \right\} \right) \quad (12)$$

Let $LS_1(p)$ and $LS_2(p)$ be any two PLTSs, λ, λ_1, $\lambda_2 \geq 0$, where ${LS_1}^{(k)}$ and ${LS_2}^{(k)}$ are the kth LTs in $LS_1(p)$ and $LS_2(p)$, respectively, and $p_1^{(k)}$ and $p_1^{(k)}$ are the probabilities of the kth LTs in $LS_1(p)$ and $LS_2(p)$, respectively. Then

$$LS_1(p) \oplus LS_2(p) = LS_2(p) \oplus LS_1(p) \quad (13)$$

$$\lambda(LS_1(p) \oplus LS_2(p)) = \lambda LS_2(p) \oplus \lambda LS_1(p) \quad (14)$$

$$(\lambda_1 + \lambda_2) LS_1(p) = \lambda_1 LS_1(p) \oplus \lambda_2 LS_1(p) \quad (15)$$

$$LS_1(p) \otimes LS_2(p) = LS_2(p) \otimes LS_1(p) \quad (16)$$

$$(LS_1(p) \otimes LS_2(p))^\lambda = LS_2(p)^\lambda \otimes LS_1(p)^\lambda \quad (17)$$

$$LS_1(p)^{(\lambda_1 + \lambda_2)} = LS_1(p)^{\lambda_1} \otimes LS_1(p)^{\lambda_2} \quad (18)$$

MSM Operator

The traditional MSM operator is given as follows:

Definition 10 [24] Suppose $a_i (i = 1, 2, \cdots, n)$ is a collection of nonnegative real numbers; then, the MSM operator is defined as

$$MSM^{(k)}(a_1, a_2, \cdots, a_n) = \left(\frac{\sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} a_{i_j}}{C_n^k} \right)^{1/k} \quad (19)$$

where $k \in (1, 2, \cdots, n)$, $(i_1, i_2, \cdots, i_k)$ traverses all $k-tuple$ combinations of $(1, 2, \cdots, n)$ and C_n^k is the binomial coefficient.

It is clear that the MSM operator has some properties as follows.

(1) $MSM^{(k)}(0, 0, \cdots, 0) = 0$, $MSM^{(k)}(a, a, \cdots, a) = a$;
(2) $MSM^{(k)}(a_1, a_2, \cdots, a_n) \le MSM^{(k)}(b_1, b_2, \cdots, b_n)$, if $a_i \le b_i$ for all i;

$$\min_i \{a_i\} \le MSM^{(k)}(a_1, a_2, \cdots, a_n) \le \max_i \{a_i\}.$$

PLMSM Operators

In this section, based on the new operational rules of PLTSs, we propose the PLMSM operator and WPLMSM operator, and then we will explore their properties and special cases.

PLMSM Operator

Definition 11 Let $LS_i(p) = \left\{ LS_i^{(t)}\left(p_i^{(t)}\right) \mid t = 1, 2, ..., \#LS_i(p) \right\} (i = 1, 2, \cdots, n)$ be n PLTSs, where $LS_i^{(k)}$ and $p_i^{(k)}$ are the kth LT and its probability, respectively, in $LS_i(p)$, and

$$PLMSM^{(k)}(LS_1(p), LS_2(p), ..., LS_n(p)) = \left(\frac{\oplus_{1 \le i_1 < \cdots < i_k \le n} \left(\otimes_{j=1}^{k} LS(p)_{i_j} \right)}{C_n^k} \right)^{\frac{1}{k}} \quad (20)$$

where $k \in (1, 2, \cdots, n)$, $(i_1, i_2, \cdots, i_k)$ traverses all $k-tuple$ combinations of $(1, 2, \cdots, n)$ and C_n^k is the binomial coefficient; the $PLMSM^{(k)}$ is called the PLMSM operator.

Theorem 1 Let $LS_i(p) = \left\{ LS_i^{(t)}\left(p_i^{(t)}\right) \mid t = 1, 2, ..., \#LS_i(p) \right\} (i = 1, 2, \cdots, n)$ be n PLTSs; then, the aggregated result by Definition 11 is expressed as

$$\begin{aligned} PLMSM^{(k)}(LS_1(p), LS_2(p), ..., LS_n(p)) &= \left(\frac{\oplus_{1 \le i_1 < \cdots < i_k \le n} \left(\otimes_{j=1}^{k} LS(p)_{i_j} \right)}{C_n^k} \right)^{\frac{1}{k}} \\ &= \cup_{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), ..., \eta_n^{(t)} \in g(LS_n(p))} g^{-1} \left(\left\{ \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \eta_{i_j}^{(t)} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} p_{i_j}^{(t)} \right) \right\} \right) \end{aligned} \quad (21)$$

Proof According to the operational rules of the PLTSs, we have

$$\otimes_{j=1}^{k} LS(p)_{i_j} = \cup_{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), ..., \eta_n^{(t)} \in g(LS_n(p))} g^{-1} \left(\left\{ \left(\prod_{j=1}^{k} \eta_{i_j}^{(t)} \right) \left(\prod_{j=1}^{k} p_{i_j}^{(t)} \right) \right\} \right),$$

and

$$\oplus_{1 \le i_1 < \cdots < i_k \le n} \left(\otimes_{j=1}^{k} LS(p)_{i_j} \right) = \cup_{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), ..., \eta_n^{(t)} \in g(LS_n(p))} g^{-1} \left(\left\{ \left(1 - \prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \eta_{i_j}^{(t)} \right) \right) \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} p_{i_j}^{(t)} \right) \right\} \right).$$

Then, we obtain

$$\frac{1}{C_n^k} \left(\oplus_{1 \le i_1 < \cdots < i_k \le n} \left(\otimes_{j=1}^{k} LS(p)_{i_j} \right) \right) = \cup_{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), ..., \eta_n^{(t)} \in g(LS_n(p))} g^{-1} \left(\left\{ \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \eta_{i_j}^{(t)} \right) \right)^{\frac{1}{C_n^k}} \right) \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} p_{i_j}^{(t)} \right) \right\} \right),$$

and

$$\left(\frac{\oplus_{1 \le i_1 < \cdots < i_k \le n} \left(\otimes_{j=1}^{k} LS(p)_{i_j} \right)}{C_n^k} \right)^{\frac{1}{k}} = \cup_{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), ..., \eta_n^{(t)} \in g(LS_n(p))} g^{-1} \left(\left\{ \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \eta_{i_j}^{(t)} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} p_{i_j}^{(t)} \right) \right\} \right),$$

Therefore,

$$PLMSM^{(k)}(LS_1(p), LS_2(p), ..., LS_n(p)) = \left(\frac{\underset{1 \le i_1 < \cdots < i_k \le n}{\oplus} \left(\overset{k}{\underset{j=1}{\otimes}} LS(p)_{i_j} \right)}{C_n^k} \right)^{\frac{1}{k}}$$

$$= \underset{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), ..., \eta_n^{(t)} \in g(LS_n(p))}{\cup} g^{-1}\left(\left\{ \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{k} \eta_{i_j}^{(t)} \right) \right)^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} p_{i_j}^{(t)} \right) \right\} \right)$$

Q.E.D.

Example 1 Let $LS_1 = \{s_0(1)\}$, $LS_2 = \{s_{-2}(0.8)\}$ and $LS_3 = \{s_1(0.4), s_2(0.6)\}$ be three PLTSs; by the function g, we can obtain $g(LS_1) = \{0.5(1)\}$, $g(LS_2) = \{0(0.8)\}$ and $g(LS_3) = \{0.75(0.4), 1(0.6)\}$, and then we can obtain the aggregation result by the *PLMSM* operator shown as follows.

Without loss of generality, suppose $k = 2$; then, according to (21), we obtain

$$PLMSM^{(2)}(LS_1, LS_2, LS_3) = \left(\frac{\underset{1 \le i_1 < \cdots < i_2 \le 3}{\oplus} \left(\overset{2}{\underset{j=1}{\otimes}} LS_{i_j} \right)}{C_3^2} \right)^{\frac{1}{2}}$$

$$= \underset{\eta_1^{(t)} \in g(LS_1), \eta_2^{(t)} \in g(LS_2), \eta_3^{(t)} \in g(LS_3)}{\cup} g^{-1}\left(\left\{ \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{2} \eta_{i_j}^{(t)} \right) \right)^{\frac{1}{C_3^2}} \right)^{\frac{1}{2}} \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{2} p_{i_j}^{(t)} \right) \right\} \right) = \{s_{-0.48}(0.256), s_{-0.18}(0.384)\}.$$

Thus, $PLMSM^{(2)}(LS_1, LS_2, LS_3) = \{s_{-0.48}(0.256), s_{-0.18}(0.384)\}$.

Further, we shall investigate some desirable properties of the *PLMSM* operator.

Theorem 2 (Commutativity) If $\left\{LS_1'(p), LS_2'(p), ..., LS_n'(p)\right\}$ is any permutation of $\{LS_1(p), LS_2(p), ..., LS_n(p)\}$, then

$$PLMSM^{(k)}\left(LS_1'(p), LS_2'(p), ..., LS_n'(p)\right) = PLMSM^{(k)}(LS_1(p), LS_2(p), ..., LS_n(p)) \tag{22}$$

Proof

$$PLMSM^{(k)}(LS_1(p), LS_2(p), ..., LS_n(p)) = \left(\frac{\underset{1 \le i_1 < \cdots < i_k \le n}{\oplus} \left(\overset{k}{\underset{j=1}{\otimes}} LS(p)_{i_j} \right)}{C_n^k} \right)^{\frac{1}{k}}$$

$$= \left(\frac{\underset{1 \le i_1 < \cdots < i_k \le n}{\oplus} \left(\overset{k}{\underset{j=1}{\otimes}} LS'(p)_{i_j} \right)}{C_n^k} \right)^{\frac{1}{k}} = PLMSM^{(k)}\left(LS_1'(p), LS_2'(p), ..., LS_n'(p)\right),$$

which completes the proof of the Theorem 2. Q.E.D.

Then, we will explore some special properties of the PLMSM operator regarding parameter k.

(1) When $k = 1$, the PLMSM operator will reduce to the PLA (probabilistic linguistic averaging) operator, i.e.,

$$PLMSM^{(1)}(LS_1(p), LS_2(p), ..., LS_n(p)) = \underset{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), ..., \eta_n^{(t)} \in g(LS_n(p))}{\cup} g^{-1}\left(\left\{ \left(1 - \left(\prod_{i=1}^{n} \left(1 - \eta_{i_j}^{(t)} \right) \right)^{\frac{1}{n}} \right) \left(\prod_{i=1}^{n} p_{i_j}^{(t)} \right) \right\} \right) \tag{23}$$

(2) When $k = 2$, the PLMSM operator will reduce to the PLBM (probabilistic linguistic Bonferroni mean) operator ($p = 1, q = 1$), i.e.,

$$PLMSM^{(2)}(LS_1(p), LS_2(p), ..., LS_n(p)) = \underset{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), ..., \eta_n^{(t)} \in g(LS_n(p))}{\cup} g^{-1}\left(\left\{ \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{2} \eta_{i_j}^{(t)} \right) \right)^{\frac{1}{C_n^2}} \right)^{\frac{1}{2}} \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{2} p_{i_j}^{(t)} \right) \right\} \right) \tag{24}$$

(3) When $k = n$, the PLMSM operator will reduce to the PLG (probabilistic linguistic geometric) operator, i.e.,

$$PLMSM^{(n)}(LS_1(p), LS_2(p), \ldots, LS_n(p)) = \bigcup_{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), \ldots, \eta_n^{(t)} \in g(LS_n(p))} g^{-1}\left(\left\{\left(\prod_{j=1}^{n} \eta_{i_j}^{(t)}\right)^{\frac{1}{n}} \left(\prod_{j=1}^{n} p_{i_j}^{(t)}\right)\right\}\right) \tag{25}$$

The WPLMSM Operator

Definition 12 Let $LS_i(p) = \left\{LS_i^{(t)}\left(p_i^{(t)}\right) \mid t = 1, 2, ..., \#LS_i(p)\right\}$ $(i = 1, 2, ..., n)$ be n PLTSs and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $LS_i(p)$ with $\omega_i \in [0, 1]$, $i = 1, 2, ..., n$, and $\sum_{i=1}^{n} \omega_i = 1$. If

$$WPLMSM^{(k)}(LS_1(p), LS_2(p), ..., LS_n(p)) = \left(\frac{\bigoplus_{1 \le i_1 < \cdots < i_k \le n}\left(\bigotimes_{j=1}^{k}\left(\omega_{i_j} \otimes LS(p)_{i_j}\right)\right)}{C_n^k}\right)^{\frac{1}{k}} \tag{26}$$

then WPLMSM is called the WPLMSM operator, where $k \in (1, 2, \cdots, n)$, $(i_1, i_2, ..., i_k)$ traverses all $k-tuple$ combinations of $(1, 2, ..., n)$, and C_n^k is the binomial coefficient.

Further, we can derive the following theorem.

Theorem 3 Let $LS_i(p) = \left\{LS_i^{(t)}\left(p_i^{(t)}\right) \mid t = 1, 2, ..., \#LS_i(p)\right\}$ $(i = 1, 2, ..., n)$ be n PLTSs and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $LS_i(p)$ with $\omega_i \in [0, 1]$, $i = 1, 2, ..., n$, and $\sum_{i=1}^{n} \omega_i = 1$. Then

$$WPLMSM^{(k)}(LS_1(p), LS_2(p), \ldots, LS_n(p)) = \bigcup_{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), \ldots, \eta_n^{(t)} \in g(LS_n(p))} g^{-1}\left(\left\{\left(1-\left(\prod_{1 \le i_1 < \cdots < i_k \le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\eta_{i_j}^{(t)}\right)^{\omega_{i_j}}\right)\right)\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} p_{i_j}^{(t)}\right)\right\}\right) \tag{27}$$

The proof of this theorem is similar to Theorem 1, so it is omitted here.

Example 2 Let $LS_1 = \{s_0(1)\}$, $LS_2 = \{s_{-2}(0.8)\}$, and $LS_3 = \{s_1(0.4), s_2(0.6)\}$ be three PLTSs; by the function g, we can obtain $g(LS_1(p)) = \{0.5(1)\}$, $g(LS_2(p)) = \{0(0.8)\}$, and $g(LS_3(p)) = \{0.75(0.4), 1(0.6)\}$. Suppose $\omega = (0.4, 0.32, 0.28)^T$ is the weight vector of $LS_i (i = 1, 2, 3)$; then, we can use the *WPLMSM* operator to aggregate three PLTSs and obtain (without loss of generality, suppose $k = 2$)

Table 1 The decision matrix with PLTSs

	C_1	C_2	C_3	C_4	C_5
A_1	$LS_{11}(p)$	$LS_{12}(p)$	$LS_{13}(p)$	$LS_{14}(p)$	$LS_{15}(p)$
A_2	$LS_{21}(p)$	$LS_{22}(p)$	$LS_{23}(p)$	$LS_{24}(p)$	$LS_{25}(p)$
A_3	$LS_{31}(p)$	$LS_{32}(p)$	$LS_{33}(p)$	$LS_{34}(p)$	$LS_{35}(p)$
A_4	$LS_{41}(p)$	$LS_{42}(p)$	$LS_{43}(p)$	$LS_{44}(p)$	$LS_{45}(p)$

$LS_{11}(p) = \{s_{-2}(0.03), s_{-1}(0.1), s_0(0.37), s_1(0.18), s_2(0.32)\}$, $LS_{12}(p) = \{s_{-2}(0.02), s_{-1}(0.03), s_0(0.17), s_1(0.21), s_2(0.58)\}$,
$LS_{13}(p) = \{s_{-2}(0.04), s_{-1}(0.04), s_0(0.13), s_1(0.22), s_2(0.57)\}$, $LS_{14}(p) = \{s_{-2}(0.02), s_{-1}(0.04), s_0(0.24), s_1(0.17), s_2(0.53)\}$,
$LS_{15}(p) = \{s_{-2}(0.15), s_{-1}(0.12), s_0(0.22), s_1(0.18), s_2(0.33)\}$, $LS_{21}(p) = \{s_0(0.12), s_1(0.3), s_2(0.58)\}$,
$LS_{22}(p) = \{s_0(0.03), s_1(0.21), s_2(0.76)\}$, $LS_{23}(p) = \{s_{-1}(0.03), s_1(0.52), s_2(0.45)\}$,
$LS_{24}(p) = \{s_{-2}(0.06), s_0(0.03), s_1(0.24), s_2(0.67)\}$, $LS_{25}(p) = \{s_{-1}(0.06), s_0(0.15), s_1(0.39), s_2(0.39)\}$,
$LS_{31}(p) = \{s_{-2}(0.05), s_{-1}(0.11), s_0(0.16), s_1(0.35), s_2(0.32)\}$, $LS_{32}(p), = \{s_0(0.08), s_1(0.27), s_2(0.65)\}$,
$LS_{33}(p) = \{s_{-1}(0.03), s_0(0.11), s_1(0.19), s_2(0.68)\}$, $LS_{34}(p) = \{s_0(0.03), s_1(0.35), s_2(0.62)\}$
$LS_{35}(p) = \{s_{-2}(0.08), s_{-1}(0.16), s_0(0.19), s_1(0.35), s_2(0.22)\}$, $LS_{41}(p) = \{s_{-2}(0.05), s_{-1}(0.05), s_0(0.14), s_1(0.33), s_2(0.44)\}$,
$LS_{42}(p) = \{s_{-2}(0.05), s_{-1}(0.07), s_0(0.14), s_1(0.26), s_2(0.49)\}$, $LS_{43}(p) = \{s_{-2}(0.02), s_0(0.12), s_1(0.07), s_2(0.79)\}$,
$LS_{44}(p) = \{s_{-2}(0.05), s_{-1}(0.07), s_0(0.14), s_1(0.19), s_2(0.56)\}$, $LS_{45}(p) = \{s_{-2}(0.02), s_{-1}(0.05), s_0(0.12), s_1(0.21), s_2(0.6)\}$

$$WPLMSM^{(2)}(LS_1, LS_2, LS_3) = \left(\frac{\underset{1 \le i_1 < i_2 \le 3}{\oplus} \left(\overset{2}{\underset{j=1}{\otimes}} \left(\omega_{i_j} \otimes LS_{i_j} \right) \right)}{C_3^2} \right)^{\frac{1}{2}} = \underset{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), \eta_3^{(t)} \in g(LS_3(p))}{\cup} .$$

$$g^{-1}\left(\left\{ \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{2} \left(1 - \left(1 - \eta_{i_j}^{(t)} \right)^{\omega_{i_j}} \right) \right) \right)^{\frac{1}{C_3^2}} \right)^{\frac{1}{2}} \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{2} p_{i_j}^{(t)} \right) \right\} \right) = \{ s_{-1.35}(0.256), s_{-0.85}(0.384) \}$$

Based on the operational rules of the PLTSs, the WPLMSM operator has also the property of commutativity described as follows:

Theorem 4 (Commutativity) Let $\left\{ LS_1'(p), LS_2'(p), ..., LS_n'(p) \right\}$ be any permutation of $\{LS_1(p), LS_2(p), ..., LS_n(p)\}$; then

$$WPLMSM^{(k)}\left(LS_1'(p), LS_2'(p), ..., LS_n'(p) \right) = WPLMSM^{(k)}(LS_1(p), LS_2(p), ..., LS_n(p)) \quad (28)$$

Then, we will discuss some special properties of the WPLMSM operator regarding parameter k.

(1) When $k = 1$, the WPLMSM operator will reduce to the PLWA (probabilistic linguistic weighted averaging) operator, i.e.,

$$WPLMSM^{(1)}(LS_1(p), LS_2(p), ..., LS_n(p)) = \underset{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), ..., \eta_n^{(t)} \in g(LS_n(p))}{\cup} g^{-1} \left(\left\{ \left(1 - \left(\prod_{i=1}^{n} \left(1 - \eta_{i_j}^{(t)} \right)^{\omega_{i_j}} \right)^{\frac{1}{n}} \right) \left(\prod_{i=1}^{n} p_{i_j}^{(t)} \right) \right\} \right) \quad (29)$$

(2) When $k = 2$, the WPLMSM operator will reduce to the PLWBM (probabilistic linguistic weighted Bonferroni mean) operator ($p = 1, q = 1$), i.e.,

$$WPLMSM^{(2)}(LS_1(p), LS_2(p), ..., LS_n(p)) = \underset{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), ..., \eta_n^{(t)} \in g(LS_n(p))}{\cup} g^{-1} \left(\left\{ \left(1 - \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(1 - \prod_{j=1}^{2} \left(1 - \left(1 - \eta_{i_j}^{(t)} \right)^{\omega_{i_j}} \right) \right) \right)^{\frac{1}{C_n^2}} \right)^{\frac{1}{2}} \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{2} p_{i_j}^{(t)} \right) \right\} \right) \quad (30)$$

(3) When $k = n$, the WPLMSM operator will reduce to the PLWG (probabilistic linguistic weighted geometric) operator, i.e.,

$$WPLMSM^{(n)}(LS_1(p), LS_2(p), ..., LS_n(p)) = \underset{\eta_1^{(t)} \in g(LS_1(p)), \eta_2^{(t)} \in g(LS_2(p)), ..., \eta_n^{(t)} \in g(LS_n(p))}{\cup} g^{-1} \left(\left\{ \left(\prod_{j=1}^{n} \left(1 - \left(1 - \eta_{i_j}^{(t)} \right)^{\omega_{i_j}} \right) \right)^{\frac{1}{n}} \left(\prod_{j=1}^{n} p_{i_j}^{(t)} \right) \right\} \right) \quad (31)$$

To show how to obtain the PLI and use the WPLMSM operator, we can provide an example.

Example 3 Recently, a friend wants to buy a new smart phone from a website, and he hesitates with four alternatives, including the APPLE iPhone X (A_1), HUAWEI P20 (A_2), Samsung GALAXY S9 (A_3), and XIAOMI MIX 2S (A_4). To make a better decision, the online comment information on the website needs to be considered. By browsing the online comment information of the four alternatives above from the Zol website (Zol.com.cn), the following five attributes associated with the alternatives are considered: endurance (C_1), photography (C_2), performance (C_3), appearance (C_4), and cost performance (C_5). According to my friend's request, the attribute weight vector is set as $\omega = (0.05, 0.15, 0.25, 0.2, 0.35)^T$. In addition, the scale of the rating on the website is $B = \{2, 4, 6, 8, 10\}$ = {low, slightly lower, medium, slightly higher, high}. For the sake of convenience, we change the scale of the rating in the website into $S = \{s_{-2}, s_{-1}, s_0, s_1, s_2\}$ = {low, slightly lower, medium, slightly higher, high}.

From this real decision problem, we can see that the five attributes mentioned are related to each other. Hence, we need to use a decision method that can consider the interrelationship between the attributes, and the probabilistic linguistic can accurately express the online comment information. Therefore, the method we proposed in this paper can solve this decision problem properly (Table 1).

Table 2 The decision matrix with PLTSs for Example 4

	C_1	C_2	C_3	C_4
A_1	$\{s_0(1)\}$	$\{s_{-2}(0.6)\}$	$\{s_1(0.4), s_2(0.4)\}$	$\{s_1(0.8)\}$
A_2	$\{s_0(0.8)\}$	$\{s_{-1}(0.8)\}$	$\{s_{-2}(0.6), s_{-1}(0.2)\}$	$\{s_0(0.6)\}$
A_3	$\{s_{-1}(0.4), s_{-2}(0.6)\}$	$\{s_2(0.6), s_1(0.2)\}$	$\{s_0(0.8)\}$	$\{s_{-2}(0.5), s_{-1}(0.5)\}$
A_4	$\{s_{-1}(0.8)\}$	$\{s_{-2}(0.6)\}$	$\{s_2(0.5), s_1(0.5)\}$	$\{s_0(1)\}$

We convert the original online comment data into the form of PLTSs. Then, we obtain the probabilistic linguistic decision matrix of four alternatives, shown as follows:

Then, we calculate the aggregated value for all attributes of each alternative by the WPLMSM operator (suppose $k = 5$) and obtain

$$LS_i = WPLMSM^{(5)}(LS_{i1}(p), LS_{i2}(p), LS_{i3}(p), LS_{i4}(p), LS_{i5}(p)) = \bigcup_{\eta_{i1}^{(t)}\in g(LS_{i1}(p)),\eta_{i2}^{(t)}\in g(LS_{i2}(p)),\ldots,\eta_{i5}^{(t)}\in g(LS_{i5}(p))} g^{-1}\left(\left\{\left(\prod_{j=1}^{5}\left(1-\left(1-\eta_{i_j}^{(t)}\right)^{\omega_{i_j}}\right)\right)^{\frac{1}{5}}\left(\prod_{j=1}^{5}p_{i_j}^{(t)}\right)\right\}\right).$$

Based on the aggregated values, we calculate the score functions for each $A_i(i = 1, 2, 3, 4)$ by Formula (3) and obtain

$$E(LS_1) = s_{-0.87}, E(LS_2) = s_{0.005}, E(LS_3) = s_{-0.538}, E(LS_4) = s_{-0.433}.$$

According to the Definitions 5–7, the ranking is $A_2 \succ A_4 \succ A_3 \succ A_1$; thus, the best choice is HUAWEI P20.

A Decision-Making Method Based on the WPLMSM Operator

For a MADM problem with PLI, let $A = \{A_1, A_2, ..., A_m\}$ be a finite set of alternatives, $C = \{C_1, C_2, ..., C_n\}$ be the set of attributes and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ `be the weight vector of attributes $C_j (j = 1, 2, ..., n)$, with $\omega_j \in [0, 1] j = 1, 2, ..., n$ and $\sum_{j=1}^{n}\omega_j = 1$. Suppose that $R = [LS(p)_{ij}]_{m\times n}$ is the decision matrix, where $LS(p)_{ij} = \left\{LS_{ij}^{(t)}\left(p_{ij}^{(t)}\right)|t = 1, 2, ..., \#LS(p)_{ij}\right\}$ is a PLTS, which is an evaluation value of alternative A_i regarding attribute C_j. The goal of this MADM problem is to rank the alternatives and select the best one.

In the following, we will use the WPLMSM operator to rank the alternatives, and the steps are shown as follows:

Step 1. Normalize the attribute values

In real decision-making, the attribute values have two types, i.e., cost type and benefit type. To eliminate the difference in types, we need to convert them to the same type.

We can convert the cost type into a benefit type, and the transformed decision matrix is expressed by $R = [LS(p)_{ij}]_{m\times n}$, where

$$LS(p)_{ij} = \begin{cases} \left\{LS_{ij}^{(t)}\left(p_{ij}^{(t)}\right)|t = 1, 2, ..., \#LS(p)_{ij}\right\} & \text{for benefit attribute } C_j \\ \left\{-LS_{ij}^{(t)}\left(p_{ij}^{(t)}\right)|t = 1, 2, ..., \#LS(p)_{ij}\right\} & \text{for cost attribute } C_j \end{cases} \quad (32)$$

Step 2. Calculate the aggregated value for all attributes of each alternative as follows.

$$LS_i = WPLMSM\left(LS(p)_{i1}, LS(p)_{i2}, ..., LS(p)_{in}\right) \quad (33)$$

where $i = 1, 2, ..., m$.

Step 3. Rank $LS_i (i = 1, 2, ..., m)$ according to Definition 7.
Step 4. End.

Table 3 The transformed decision matrix by g function for Example 4

	C_1	C_2	C_3	C_4
A_1	{0.5(1)}	{0(0.6)}	{0.75(0.4), 1(0.4)}	{0.75(0.8)}
A_2	{0.5(0.8)}	{0.25(0.8)}	{0(0.6), 0.25(0.2)}	{0.5(0.6)}
A_3	{0.25(0.4), 0(0.6)}	{1(0.6), 0.75(0.2)}	{0.5(0.8)}	{0(0.5), 0.25(0.5)}
A_4	{0.25(0.8)}	{0(0.6)}	{1(0.5), 0.75(0.5)}	{0.5(1)}

Table 4 The ranking results for the different values of parameter k for Example 4

	$E(LS_1)$	$E(LS_2)$	$E(LS_3)$	$E(LS_4)$	Ranking
$k=1$	$s_{0.7116}$	$s_{0.7079}$	$s_{0.2474}$	$s_{0.2350}$	$A_1 \succ A_2 \succ A_3 \succ A_4$
$k=2$	$s_{-0.9799}$	$s_{-1.0034}$	$s_{-1.4571}$	$s_{-1.3488}$	$A_1 \succ A_2 \succ A_4 \succ A_3$
$k=3$	$s_{-1.5635}$	$s_{-1.6982}$	$s_{-1.8091}$	$s_{-1.8326}$	$A_1 \succ A_2 \succ A_4 \succ A_3$
$k=4$	$s_{-1.7228}$	$s_{-1.7407}$	$s_{-2.0766}$	$s_{-2.1078}$	$A_1 \succ A_2 \succ A_4 \succ A_3$

Numerical Example

This section presents a risk assessment example about the hot topic "Belt and Road" to illustrate the proposed method (adapted from [46]).

Example 4 In the China-Central Asia-West Asia economic corridor, because China advocates for the "B&R" strategy, the Central and West Asian countries strengthen the investment cooperation with Chinese companies in energy, infrastructure, and other fields. This brings opportunity but also increases risk. A Chinese company is planning to invest in a country along the China-Central Asia-West Asia economic corridor. It aims to choose a country of the lowest risk from the four alternative countries: United Arab Emirates (UAE), Saudi Arabia, Qatar, and Israel. A group of experts (including five experts) were invited to provide their preference information over the alternative countries by comparing each pair of them. Because of the complexity of the evaluation problem, the experts cannot give specific risk assessment values in crisp numbers but rather in the form of LTs. Furthermore, the experts may be hesitant among many LTs, and they may have different degrees of preference for these countries because their knowledge and the information are not equivalent. Therefore, using PLTSs to express preference information is reasonable and necessary.

For this decision-making problem, let $x_i(i=1,2,3,4)$ represent UAE, Saudi Arabia, Qatar, and Israel, respectively. The experts compare each pair of countries using the linguistic evaluation scale $S=\{s_{-2}, s_{-1}, s_0, s_1, s_2\}$ = {low, slightly lower, medium, slightly higher, high}. The preference information given by the experts is shown in Tables 2 and 3.

The Decision Steps

Step 1: Because all attributes are of the benefit type, there is no need to normalize the attribute values.

Step 2: Calculate the aggregated value for all attributes of each alternative by Formula (33) and suppose $k=2$.

Because there are too much data, they are omitted here (please refer to the Excel data file).

Step 3: Calculate the score function for each $LS_i(i=1,2,3,4)$ based on the Formula (3), and we have

$$E(LS_1) = s_{-0.9799}, E(LS_2) = s_{-1.0034}, E(LS_3) = s_{-1.4571}, E(LS_4) = s_{-1.3488}.$$

Step 4: Rank the alternatives. According to Definition 4, the ranking is $A_1 \succ A_2 \succ A_4 \succ A_3$.

Then, we explain the influence on the ranking results for the different values of parameter k, and the ranking results are shown in Table 4.

From Table 4, we can see that the rankings of the alternatives vary with parameter k. When $k=1$, the ranking result is different from the others. Clearly, this is reasonable because the situation of $k=1$ does not consider the interrelationship among the attributes while the others consider the interrelationships between two attributes or among three or four attributes. In addition, from Table 4, we also know the score functions are becoming increasingly smaller as parameter k increases, so we can regard parameter k as a decision-maker's risk preference. The smaller parameter k is, the more the decision-maker is risk-seeking; the bigger parameter k is, the greater the risk aversion.

Discussion

To demonstrate the validity of the proposed method, we can compare it with the method proposed by Bai et al. [4], which is

Table 5 Comparison using two methods

Method	Aggregation operator	Score functions $E(LH_i)$	Ranking
Bai et al. [4]	*PLWA*	$E(LH_1) = s_{1.924}, E(LH_2) = s_{2.744}, E(LH_3) = s_{2.743}, E(LH_4) = s_{2.656}$	$H_2 \succ H_3 \succ H_4 \succ H_1$
Proposed method	*WPLMSM*($k=1$)	$E(LH_1) = s_{1.924}, E(LH_2) = s_{2.744}, E(LH_3) = s_{2.743}, E(LH_4) = s_{2.656}$	$H_2 \succ H_3 \succ H_4 \succ H_1$

Table 6 Ranking results by different methods

Method	Aggregation operator	Expected values $E(LH_i)$	Ranking
Bai et al. [4]	*PLWA*	$E(LH_1)=s_{1.924}, E(LH_2)=s_{2.744}, E(LH_3)=s_{2.743}, E(LH_4)=s_{2.656}$	$H_2 \succ H_3 \succ H_4 \succ H_1$
Proposed method	*WPLMSM*$(k=2)$	$E(LH_1)=s_{-1.672}, E(LH_2)=s_{-0.859}, E(LH_3)=s_{-1.085}, E(LH_4)=s_{-0.683}$	$H_4 \succ H_2 \succ H_3 \succ H_1$

based on the PLWA operator. Because the PLWA operator does not consider the interrelationship among the attributes, we can compare it with our method when $k=1$.

Example 5 Because of limited medical resources and the increasingly serious environmental pollution in China, it is necessary to evaluate large domestic hospitals to search for the optimal hospital with appropriate resource allocation and reasonable resource input and output. Three attributes are adopted: the hospital environmental status (C_1), personalized diagnosis and treatment optimization (C_2), and social resource allocation optimization (C_3). The weight vector of these attributes is $\omega=(0.2, 0.1, 0.7)^T$. The four hospitals to be evaluated are the West China Hospital of Sichuan University (H_1), the Huashan Hospital of Fudan University (H_2), the Union Medical College Hospital (H_3), and the Chinese PLA General Hospital (H_4). The ranking results are shown in Table 5.

From Table 5, we can find not only the same ranking results but also the same score functions for the same hospital by the two methods. The reason is that when $k=1$, the proposed method reduces to the PLWA operator, and the two methods use the same operational laws. Therefore, the ranking results are exactly the same. Hence, the method proposed in this paper is effective and feasible.

To demonstrate the advantages of the proposed method in this paper, we use the same example to provide a new comparison listed in Table 6.

From Table 6, we find different ranking results using the two methods. From Example 5, we can see that the attributes are intrinsically related, but the method of Bai et al. [4] does not take this interrelationship into account, which will lead to an unreasonable result. To clarify, we further provide a detailed comparison.

Compared with the method based on the PLWA operator proposed by Bai et al. [4], the proposed method adopts the same new operational rules. However, the proposed method integrates the MSM operator which has the remarkable feature of capturing the interrelationships among multiple attributes. Thus, the proposed method can better handle the relationship among input variables. The method of Bai et al. [4] uses the WA operator that does not take account the relationship between the input arguments, so when the relationship of the inputs needs to be considered, it will produce an unreasonable ranking result. In addition, the parameter k in the proposed method can be regarded as the risk preference of decision-makers. Clearly, the proposed method is more flexible and general for solving MADM problems than the method proposed by Bai et al. [4].

Further, we can provide another a comparison based on the PLWA operator proposed by Zhang et al. [46]. The operational rules adopted by the PLWA operator proposed by Zhang et al. [46] are different from those used in our proposed method. The ranking results are shown in Table 7.

From Table 7, we find different ranking results using the two methods. When $k=1$, the proposed method reduces to the PLWA operator, but in this example the two methods use the different operational laws. The method of Zhang et al. [46] adopts the operational laws proposed by Pang et al. [26], and the method in this paper adopts the operational laws proposed by Gou [9]. In summary, the comparison results can be briefly summed up as follows.

Compared with the method based on the PLWA proposed by Zhang et al. [46], the proposed method adopts the new operational rules with the accurate operations, while the method proposed by Zhang et al. [46] only adopts the simple traditional operational rules. However, the traditional operational laws have some shortcomings in processing the LTs. Furthermore, the new operational laws can not only avoid the operational results exceeding the bounds of LTSs but also retain the complete probability information after operations. This aspect supports the proposed method as being more reasonable and accurate than the method of Zhang et al. [46].

Table 7 Ranking results by different methods

Method by	Aggregation operator	Expected values $E(LS_i)$	Ranking
Zhang et al. [46]	*PLWA*	$E(LS_1)=s_{0.0413}, E(LS_2)=s_{0.2195}, E(LS_3)=s_{-0.1176}, E(LS_4)=s_{-0.1176}$	$A_2 \succ A_1 \succ A_3 \succ A_4$
Proposed method	*WPLMSM*$(k=1)$	$E(LS_1)=s_{0.7116}, E(LS_2)=s_{0.7079}, E(LS_3)=s_{0.2474}, E(LS_4)=s_{0.2350}$	$A_1 \succ A_2 \succ A_3 \succ A_4$

Table 8 Comparison of two methods

Method by	Aggregation operator	Ranking
Gou [9]	*HFLWA*	$A_3 \succ A_2 \succ A_4 \succ A_1$
Proposed method	*WPLMSM*$(k=1)$	$A_2 \succ A_3 \succ A_4 \succ A_1$

Finally, we make a comparison of our proposed method with the method based on the HFLWA operator proposed by Gou [9]. When we set $k=1$ and use the same operational rules proposed by Gou [9], the ranking results of the two methods are shown in Table 8.

From Table 8, we find different ranking results by using the two methods. We will discuss the reason for these results.

Compared with the method based on the HFLWA proposed by Gou [9], the proposed method expressed the evaluation information with the PLTS, whereas the method proposed by Gou [9] expressed the evaluation information with the HFLTSs. HFLTSs can collect some possible LTs provided by the DMs, but these LTs have equal probability; thus, they ignore the effect of the probability. The PLTS can not only contain all the possible LTs such as the HFLTSs but also express the degree of importance or weight of each LT. Thus, the PLTSs can express the evaluation information more accurately than HFLTSs in practical problems.

Conclusion

The MADM problems based on the PLI have been applied to a range of areas. Because PLI can reflect different degrees of importance or weights of all possible evaluation values, and the MSM operator has the remarkable feature of capturing the interrelationships among multiple attributes, in this paper, we extended the MSM operator to the environment of PLI and proposed some MSM aggregation operators for the PLI, such as the PLMSM operator and WPLMSM operator; some desirable properties and special cases of these operators are illustrated. Finally, a MADM method with the PLI based on the WPLMSM operator is proposed. This method is more flexible and general for solving MADM problems with PLI than the existing methods.

In further research, it is necessary to use the proposed method for some real decision-making problems, or the proposed operators should be extended to some new fuzzy information.

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Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

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