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A Multicriteria Decision-Making Approach with Linguistic D Numbers Based on the Choquet Integral

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Abstract

Linguistic D numbers (LDNs) provide a reliable expression of cognitive information. By inheriting the advantages of linguistic terms (LTs) and D numbers (DNs), LDNs can express uncertain and incomplete cognitive information in multicriteria decision-making (MCDM), and they do so better than existing methods. The TODIM (an acronym in Portuguese of interactive and multicriteria decision-making) method can consider decision experts' (DEs') bounded rationality, such as cognition toward loss, which is caused by the DEs' cognitive limitations during the decision process. Additionally, the Choquet integral can process the interrelationship among criteria or cognitive preferences, which helps to reflect the complex cognition of DEs. Therefore, it is necessary to propose a novel cognitive MCDM approach by extending the TODIM method and Choquet integral to handle MCDM problems in which the cognitive information is expressed by LDNs. In this paper, we introduced LDNs to represent uncertain and hesitant cognitive information. The definition and comparison approach of LDNs were also recommended. Then, we proposed the distance function and modified the score function of LDNs. Later, considering the limitations of the DEs' cognitive abilities in real decision-making and the phenomenon where attributes or cognitive preferences in MCDM problems are not independent, we developed a novel cognitive MCDM approach with LDNs by extending the TODIM method and the Choquet integral to deal with these cases. The proposed approach can not only take the influence of the limited cognitive abilities of DEs on the decision-making results into account but can also deal with the correlation between the cognitive preferences. A novel cognitive MCDM approach with LDNs based on the TODIM method and Choquet integral was proposed. Moreover, the validity and superiority of the presented approach were verified by dealing with practical problems and comparing them to other approaches. The proposed approach can consider cases where the DEs are rationally bounded in their cognitive decision-making and the criteria or cognitive preferences in MCDM problems have an interrelationship. Therefore, this approach can produce more reliable decision-making results than some existing MCDM approaches.

Keywords Multicriteria decision-making (MCDM) · Linguistic D numbers (LDNs) · Choquet integral · TODIM

Introduction

Cognition and decision-making are inextricably linked, and decision ability is one of the main characteristics of cognitive computation [31]. Multicriteria decision-making (MCDM) requires decision experts (DEs) to evaluate all available alternatives based on their own cognitive abilities and then select the most satisfactory one(s) according to the calculation results of the evaluation values. In general, it is difficult for DEs to express their evaluations on alternatives with precise values because of their cognition limitations. Considering cognitive ambiguity and indeterminacy, Liu and Li [19] used interval-valued intuitionistic fuzzy numbers (IVIFNs) (an extension of intuitionistic fuzzy numbers (IFNs) [1]) to express the indeterminacy of DEs' cognitions. Ye [39] defined the concept of hesitant neutrosophic linguistic numbers (HNLNs) to deal with DEs' cognitive hesitance. Wang et al. [36] introduced the linguistic Z numbers (LZNs) to model the cognitive information in real life. In addition, the interval-transformed hesitant fuzzy elements (ITHFEs) [7], picture fuzzy linguistic numbers (PFLNs) [22], and D-intuitionistic hesitant fuzzy sets (D-IHFSs) [17] were also extended to express the cognitive ambiguity of DEs' in MCDM.

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Some MCDM approaches [7, 17, 19, 22, 31, 36, 39] related to cognitive computing have been proposed based on the aforementioned concepts. Farhadinia [7] extended the TOPSIS (technique for order preference by similarity to ideal solution) method to handle the cognitive information expressed by ITHFEs and developed a novel cognitive MCDM approach based on the TOPSIS method. Li and Chen [17] developed a cognitive MCDM approach based on D-IHFSs. Ye [39] established some cognitive MCDM approaches by calculating the expected value and the similarity measure of HNLNs, and some other cognitive MCDM approaches were proposed on the basis of the operators of IVIFNs [19], PFLNs, and intuitionistic fuzzy values (IFVs). Among these cognitive MCDM approaches, two approaches are the most enlightening to our research. In [36], Wang et al. developed an extended TODIM (an acronym in Portuguese of interactive and MCDM) method on the basis of the Choquet integral for MCDM with LZNs. This approach considered the bounded rationality behaviors of DEs' by applying the TODIM method. Moreover, it also considered the interactivity of attributes in the process of cognitive calculation because it inherited the merits of the Choquet integral. Another approach, which was proposed by Li and Chen [17], defined the D-IHFSs to express the uncertainty, ignorance, and other important concepts in cognitive science by combining the D numbers (DNs) and hesitant fuzzy information. Inspired by these two cognitive MCDM approaches, we focus on the theories of DNs, TODIM and the Choquet integral, in this paper to present a novel MCDM approach.

The theory of DNs [3] is an extension of Dempster–Shafer evidence theory (DSET) [2], which overcomes the imperfections existing in DSET. Similar to the DSET, there are also frame of discernment (FOD) and basic probabilistic assignment (BPA) functions in DNs. However, the elements of FOD in DNs do not need to be mutually exclusive, while also releasing the completeness constraint on the degrees of confidence. Thus, DNs have a stronger ability in representing uncertain or incomplete cognitive information compared with the DSET. Due to these advantages of DNs, research on the application of DNs in the MCDM field has quickly progressed in recent years. Several researchers have presented their study points about the basic theories of DNs, such as the combination rules [4], the fusion approach for multiple DNs [34, 35], the integration values [37], and the distance function [18]. Deng and Jiang [4] defined new combination rules for DNs that concurrently consider the global and partial conflict and make up for the defect in previous studies, in which only the global conflict is considered [3]. Aimed at the shortcoming where the commutative property is not satisfied when fusing multiple DNs, Wang and Wei [35] improved the fusion approach for DNs based on the values of evaluation grades, and Wang et al. [37] developed a modified integration for DNs by redistributing the incomplete information rather than discarding it. Considering that the elements of the FOD are not required to be mutually exclusive, a novel distance function for DNs was proposed by Li et al. [18]. Furthermore, some classical ranking approaches such as the AHP (analytic hierarchy process) method [5, 6, 29, 41], the VIKOR (visekriterijumska optimizacija i kOmpromisno resenje) method [14], and the TOPSIS method [38] have also been extended to deal with DNs, thus many new MCDM approaches with DNs have been constructed.

DNs are useful tools for expressing uncertain and hesitant cognitive information in MCDM. Li and Chen [17] proposed the D-intuitionistic hesitant fuzzy sets (D-IHFSs) by combining DNs and generalized hesitant fuzzy sets (GHFSs) to express the cognitive ambiguity of DEs in MCDM. Huang [14] proposed the linguistic D numbers (LDNs) on the basis of linguistic terms (LTs) and DNs. Because of the fuzziness of human cognition, it is easier for DEs to use LTs to express cognitive information [36]. DNs can represent "uncertainty" in cognitive science well [17], so LDNs, which are the combination of DNs and LTs, can represent cognitive information more accurately in MCDM. Research on LDNs has been recently developed. In addition to giving the definition of LDNs, Huang [14] also defined the operational rules of LDNs with the linguistic score function (LSF) and proposed the prioritized weighted average operator for LDNs (LD-PWA operator) based on these operational rules. Additionally, Wang and Huang [33] put forward the fuzzy entropy of LDNs. To integrate different LDNs, Wang and Huang further [33] defined the combination rule of two LDNs by regarding the LDNs as different pieces of evidence. Moreover, they outlined an MCDM approach based on the fuzzy entropy of LDNs and the thought of evidential reasoning. It can be noted that the existing studies on LDNs are not yet mature, and not all of them consider the effect of the DEs' psychological and cognitive behaviors on the decision results, assuming that DEs are fully rational in their decision-making. However, in the actual decision-making process, DEs can only be rationally bounded because of the limitations of the DEs' cognition. Thus, it is valuable to conduct a study for MCDM problems with LDNs based on DEs' bounded rationality.

In real life, due to the influence of the DE's cognitive ability, emotion, and psychology, there is a certain deviation between the actual choice of the DEs and the optimal choice under rational decision theory. That is, DEs often have limited rational psychological behavior [27]. Therefore, studies on behavioral MCDM have become a prominent research area in the field of management science. The DEs' behaviors are often influenced by many factors, such as cognition toward loss, which is caused by the DEs' cognitive limitations in decision-making. The TODIM method [8] is a practical tool for processing MCDM problems [21, 36, 40] considering DEs' psychological and cognitive behaviors in decision-making. Many researchers have applied the TODIM method to deal with MCDM problems with diverse information. For example, Liu and Teng [21] studied an MCDM problem in which the evaluation information was denoted by 2-dimensional uncertain linguistic information. They then presented an extended TODIM approach to solve this kind of MCDM problem. Additionally, Zhang et al. [40] aimed at applying the TODIM method to the application of MCDM problems with indeterminate evaluation information and provided an extended TODIM method to process MCDM problems with neutrosophic numbers.

Most studies on MCDM are based on the assumption that attributes are independent of one another, which means that the weights of attributes satisfy the additivity property. In fact, in real decision-making, there are often interactions or correlations between attributes. Therefore, in order to better reflect the actual decision-making situation and obtain more objective decision-making results, the study of correlation measures reflecting the interaction of attributes becomes extremely important. Considering that the criteria or cognitive preferences may be influenced by one another, the Choquet integral was introduced to solve these situations [20]. Moreover, some studies [23, 30, 36] have combined the TODIM method with the Choquet integral [9, 36] to solve more complex MCDM problems. Lourenzutti et al. [23] aimed at addressing the issue that criteria are intersecting in MCDM problems using the Choquet integral and put forward two generalized MCDM approaches by extending the TODIM method to group decision-making. Tan et al. [30] created a study on uncertain MCDM problems that expands the TODIM method to solve hesitant fuzzy information. They then proposed an extended Choquet-based TODIM approach to deal with criteria interactive MCDM problems. Wang et al. [36] proposed an extended TODIM method on the basis of the Choquet integral for MCDM problems with LZNs. Peng and Yang [25] extended the Choquet integral to aggregate the Pythagorean fuzzy information and proposed two new MCDM approaches. In addition, some researchers have applied the Choquet integral to solve MCDM problems with interval gray numbers [32], interval neutrosophic uncertain linguistic variables [20], intuitionistic fuzzy probabilistic linguistic information [28], and interval neutrosophic hesitant fuzzy information [16].

The TODIM method considers the cognition toward loss of DEs in decision-making [36], the Choquet integral helps solve situations in which cognitive preferences are interactive [20] in real decisions, and the LDNs have advantages in representing uncertain and incomplete cognitive information [17, 36]. However, as far as we know, there has been no research until now that extends the TODIM method and the Choquet integral to the field of MCDM problems with LDNs. Therefore, in this paper, we will fill this gap by developing a novel cognitive MCDM approach with LDNs based on the TODIM method and the Choquet integral. Therefore, the objectives of this study are to (1) introduce the definition of LDNs and to provide DEs a convenient way to express the uncertain and hesitant cognitive information in MCDM; (2) combine the TODIM method with the Choquet integral and propose a new MCDM approach that not only applies the TODIM method and Choquet integral to the decision environment with LDNs but also considers the DEs' bounded rationality behaviors and the interaction among attributes or cognitive preferences; and (3) solve some practical MCDM problems to verify the effectiveness and superiority of the presented approach.

The framework of this paper is as follows. Section 2 reviews the concept of the LTS, the linguistic scale function (LSF), DSET, DNs, and fuzzy measure and describes the decision-making steps of the TODIM approach in the MCDM. Section 3 introduces the definition of the LDNs and proposes the distance function and the comparison approach between two LDNs. Section 4 develops a new MCDM approach with LDNs on the basis of the TODIM method and the Choquet integral. Section 5 verifies the effectiveness of the proposed approach with an illustrated example and conducts a sensitivity analysis and a comparison analysis. Section 6 describes the conclusion.

Preliminaries

In this section, we mainly introduce the concepts of linguistic term sets (LTSs), the linguistic scale function (LSF), Dempster–Shafer evidence theory (DSET), D numbers (DNs), and the fuzzy measure. In addition, the decision-making steps of the TODIM approach in MCDM problems are also briefly introduced.

Linguistic Term Sets and Linguistic Scale Function

Because of the complexity of the practical decision-making environment, it is more appropriate for DEs to use LTs to assess the alternatives. The LTs constitute an LTS. Usually, an LTS $S_t = \{s_i | i = 0| 1| 2| \cdots | 2t\}$ is a finite and completely ordered discrete set, where s_i represents a predefined LT and t is a positive integer. Generally, an LTS S_t meets the following characteristics [36]:

(1) LTS S_t is ordered: $s_i \prec s_j$ if and only if $i < j$;
(2) A negation operator is defined as:$\mathrm{neg}(s_i) = s_{2t-i}$.

When integrating LTs, the integrated values may not mismatch with the LTs in the predefined LTS. To handle this, Herrera [13] extended the discrete LTS S_t to the continuous LTS $S = \{s_\theta | \theta \in R^+\}$. If $s_\theta \in S_t$, we call s_θ an original LT; otherwise, we call s_θ an expanding LT.

To define the semantics of LTs by actual applications, LTs can be transformed into digital values by the LSF. Here, we give the definition of the LSF.

Definition 1 [24] Let $S_t = \{s_i | i = 1| 2| \cdots | 2t\}$ be a discrete LTS, and s_i be an LT. If θ_i is a digital value, then the LSF is a mapping from s_i to $\theta_i (i = 0, 1, \cdots, 2t)$ shown as follows:

$$f : s_i \rightarrow \theta_i (i = 0, 1, \cdots, 2t) \quad (1)$$

where f is a monotone increasing function. To maintain the original information as much as possible, LSF f can be extended to a strictly monotonous increasing and continuous function $\overline{f} : s_i \rightarrow \theta_i \ (s_i \in S)$. Therefore, the inverse function of $\overline{f}$ exists, and it can be marked as $\overline{f}^{-1}$.

There are many types of established LSFs based on different ideas, such as the subscript function [24] and the prospect theory [15], and each kind of LSFs has its own characteristics. Next, we introduce three types of LSFs [24]:

$$f_1(s_i) = \theta_i = \frac{i}{2t} (i = 0, 1, \cdots, 2t) \quad (2)$$

The function $f_1(s_i)$ is defined based on the subscript function, and it can equally distribute the semantic value that the LT expressed.

$$f_2(s_i) = \theta_i = \begin{cases} \dfrac{a^t - a^{t-i}}{2a^t - 2} & (i = 0, 1, \cdots, t) \\ \dfrac{a^t + a^{i-t} - 2}{2a^t - 2} & (i = t, t+1, \cdots, 2t) \end{cases} \quad (3)$$

where a is a parameter, and its value can be determined experimentally. The function $f_2(s_i)$ has a character where the absolute deviation between the semantic values of adjacent LTs increases from the middle of the given LTS to both sides.

$$f_3(s_i) = \begin{cases} \dfrac{t^\gamma - (t-i)^\gamma}{2t^\gamma} & (i = 0, 1, \cdots, t) \\ \dfrac{t^\beta + (i-t)^\beta}{2t^\beta} & (i = t, t+1, \cdots, 2t) \end{cases} \quad (4)$$

where γ, β are parameters. If $\gamma = \beta = 1$, then $f_3(s_i)$ turns to $f_1(s_i)$. The property of the function $f_3(s_i)$ is that with the extension from the middle of the given LTS to both sides, the absolute deviation between the semantic values of the adjacent LTs also decreases.

The D Numbers

The theory of DNs is developed on the basis of DSET. Therefore, in this subsection, we first introduce some basic concepts in the DSET.

Definition 2 [2] For an FOD U in which the elements are mutually exclusive, $P(U)$ represents the power set of U, and then a mass function is a mapping $m : P(U) \rightarrow [0, 1]$, which is also called a BPA, satisfying

$$\sum_{A \in P(U)} m(A) = 1 \text{ and } m(\phi) = 0 \quad (5)$$

where ϕ is an empty set, and A is a subset of $P(U)$. It is not difficult to determine that there are two restrictions in DSET: one is that it requires the elements in U to be mutually exclusive, and the other is that the sum of $m(A)$ must be equal to 1. Then, in order to make up for the shortcomings, Deng [6] proposed the concept of DNs.

Definition 3 [3] Let Ω be a finite and nonempty set, then a DN is a mapping $D : \Omega \rightarrow [0, 1]$, which is defined as follows:

$$\sum_{B \in \Omega} D(B) \leq 1, D(\phi) = 0 \quad (6)$$

where ϕ is an empty set and B is a subset of Ω. The definition of DNs differs from the DSET where the elements in set Ω do not require mutually exclusivity, while also releasing the completeness constraint on the degrees of confidence. If $\sum_{B \in \Omega} D(B) = 1$, we say this DN expresses complete information; if $\sum_{B \in \Omega} D(B) < 1$, we say this DN expresses incomplete information. Thus, DNs can better express incomplete information than the DSET.

The Fuzzy Measure

The fuzzy measure [9] is a practical tool for measuring the intersection among attributes. Its definition is described as follows:

Definition 4 [9] Let $X = \{x_1, x_2, \cdots, x_n\}$ be a fixed set, where $P(X)$ denotes the power set of X. A fuzzy measure on X is a set function $\mu : P(X) \rightarrow [0, 1]$, which satisfies the following conditions:

(1) $\mu(\phi) = 0$, $\mu(X) = 1$;
(2) If $C, D \in P(X)$ and $C \subseteq D$, then $\mu(C) < \mu(D)$.

Generally, $\mu(C)$ can be seen as the subjective magnitude level of attribute C. With the separate fuzzy measure of the criteria, the λ−fuzzy measure of any combination can be calculated by the following formula:

$$\mu(C \cup D) = \mu(C) + \mu(D) + \lambda\mu(C)\mu(D), \quad \lambda \in [-1, \infty), \quad \forall C, D \in P(X) \text{ and } C \cap D = \phi.$$

The value of the parameter λ reflects the interaction between C and D: if $\lambda = 0$, it shows that C and D are independent of one another, and μ is called the additive measure on X; if $\lambda < 0$, it shows that there is a redundant relationship between C and D, where μ is called the subadditive measure on X; if $\lambda > 0$, it shows that there is a complementary relationship between C and D, where μ is called the superadditive measure on X.

Specifically, for any subset $A \in P(X)$, the $\lambda-$ fuzzy measure of A can be obtained by formula (7):

$$\mu(A)=\begin{cases}\frac{1}{\lambda}\left(\prod_{i\in A}[1+\lambda\mu(i)]-1\right) & \lambda\neq 0\\ \sum_{i\in A}\mu(i) & \lambda=0\end{cases} \tag{7}$$

in which $\mu(i)$ represents the $\lambda-$fuzzy measure of a subset with a single element i and is usually called the fuzzy density. Since$\mu(X)$ = 1, the value of λ can be calculated by solving $\lambda+1=\prod_{i=1}^{n}[1+\lambda\mu(i)]$.

Much of the research on Choquet integrals has studied, for examples, the Choquet integral [9] and the Shapley weight [30]. The Choquet integral is often used to solve situations in which the criteria are correlative. The definition of the Choquet integral is described as follows:

Definition 5 [9] Let ψ be a real-valued function on X, and μ be a fuzzy measure on X. Then, the Choquet integral of ψ with regard to μ can be defined as:

$$C_\mu(\psi)=\sum_{i=1}^{p}\psi(x_{(i)})\left[\mu(B_{(i)})-\mu(B_{(i-1)})\right]+\sum_{i=p+1}^{n}\psi(x_{(i)})\left[\mu(A_{(i)})-\mu(A_{(i+1)})\right] \tag{8}$$

where (.) is a permutation on X so that

$$\psi(x_{(1)})\leq\psi(x_{(2)})\leq,\cdots,\leq\psi(x_{(P)})<0\leq\psi(x_{(p+1)})\leq,\cdots,\leq\psi(x_{(n)});B_{(i)}=\{x_{(1)},x_{(2)},\cdots,x_{(i)}\},B_{(0)}=\phi,\ A_{(i)}=\{x_{(i)},x_{(i+1)},\cdots,x_{(n)}\},\text{and } A_{(n+1)}=\phi. \tag{9}$$

The TODIM Method

In real life, due to the influence of the DE's cognitive ability, emotion, and psychology, there is a certain deviation between the actual choice of DEs and the optimal choice under the rational decision-making theory. That is, DEs often have limited rational psychological behavior. The TODIM method is a typical behavior decision-making method based on the prospect theory that can describe the influence of DEs' limited rational behaviors on decision-making results.

Suppose there is a set of alternatives $A = \{a_1, a_2, \cdots, a_m\}$, the criteria set is $C = \{C_1, C_2, \cdots, C_n\}$, and its weight vector is $W = \{w_1, \cdots w_j, \cdots, w_n\}$. Let $D = (x_{ij})_{m\times n}(i = 1, 2, \cdots, m, j = 1, 2, \cdots, n)$ be a decision matrix for an MCDM problem, where x_{ij} represents the attribute value given by the DEs for the alternative A_i for criterion C_j. Then, the steps to solve this MCDM problem with the TODIM method are shown as follows [30]:

Step 1: Normalize the decision matrix $D = (x_{ij})_{m\times n}$ into $R = (r_{ij})_{m\times n}$, where r_{ij} represents the normalized attribute value, and x_{ij} and r_{ij} are all crisp numbers.

Step 2: Calculate the relative weight w_{jr} of the attribute C_j to the reference attribute C_r by:

$$w_{jr}=\frac{w_j}{w_r}j,r=1,2,\cdots,n \tag{10}$$

where w_j is the weight of the attribute C_j and $w_r = max\ \{w_j\}(j = 1, 2, \cdots, n)$.

Step 3: Calculate the degree of dominance of the alternative a_i over alternative a_k with respect to criteria C_j.

$$\phi_j(a_i,a_k)=\begin{cases}\sqrt{w_{jr}(r_{ij}-r_{kj})/\sum_{j=1}^{n}w_{jr}}, & \text{if } r_{ij}-r_{kj}>0\\ 0, & \text{if } r_{ij}-r_{kj}=0\\ -\frac{1}{\theta}\sqrt{\frac{\left(\sum_{j=1}^{n}w_{jr}\right)(r_{ij}-r_{kj})}{w_{jr}}}, & \text{if } r_{ij}-r_{kj}<0\end{cases} \tag{11}$$

where r_{ij} and r_{kj} are the normalized evaluations values, and θ is the attenuation factor of the loss and denotes the degree of loss aversion of the DEs.

Step 4: Obtain the degree of dominance of alternative a_i over the alternative a_k.

$$\vartheta(a_i,a_k)=\sum_{j=1}^{n}\phi_j(a_i,a_k)i,k=1,2,\cdots,m \tag{12}$$

where $\phi_j(a_i, a_k)$ represents the contribution of criterion C_j to function $\vartheta(a_i, a_k)$.

Step 5: Obtain the global value of the alternative a_i.

$$\delta(x_i)=\frac{\sum_{k=1}^{m}\vartheta(x_i,x_k)-min_i\left\{\sum_{k=1}^{m}\vartheta(x_i,x_k)\right\}}{max_i\left\{\sum_{k=1}^{m}\vartheta(x_i,x_k)\right\}-min_i\left\{\sum_{k=1}^{m}\vartheta(x_i,x_k)\right\}},\ i=1,2,\cdots,m \tag{13}$$

Step 6: Order alternatives.

Order the alternatives by the value of $\delta(x_i)(i = 1, 2, \cdots, m)$. The optimal alternative has the highest value of $\delta(x_i)$.

Linguistic D Numbers and Related Concepts

In this section, we first introduce the concept of LDNs. Then, an extended distance function for LDNs is provided. Additionally, we improve the LDNs' score function and define the degree of deviation of the LDNs. Then, based on these a comparison approach for LDNs is defined.

Linguistic D Numbers

LDNs integrate the merits of LTs and DNs, so they can express uncertain and hesitant cognitive information in the MCDM more appropriately. Definition 6 illustrates the definition of LDNs.

Definition 6 [14] Let $S_t = \{s_i | i = 0| 1| \cdots | 2t\}$ be an LTS, then an LDN is a mapping $LD : S_t \rightarrow [0, 1]$, which is defined as follows:

$$\sum_{S' \in S_t} LD\left(S'\right) \leq 1, LD(\phi) = 0 \tag{14}$$

in which ϕ is an empty set and S' is a subset of S_t. LDNs are the special forms of DNs.

An LDN can be denoted as $LD(\{s_i\}) = v_i$, $i = 0, 1, \cdots, 2t$, which is more succinctly written as $LD = \{(s_0, v_0), (s_1, v_1), \cdots(s_i, v_i), \cdots(s_{2t}, v_{2t})\}$, in which $v_i > 0$ and $\sum_{i=0}^{2t} v_i \leq 1$. If $\sum_{i=0}^{2t} v_i = 1$, it shows the information expressed by this LDN is complete; if $\sum_{i=0}^{2t} v_i < 1$, it shows the information expressed by this LDN is incomplete.

Example 1 Suppose there is an experiment to access the performance of the iPhone X and the Galaxy Note 8, and the evaluation LTS is $S_2 = \{s_0(\textit{\textbf{extremely poor}}), s_1(\textit{\textbf{poor}}), s_2(\textit{\textbf{general}}), s_3(\textit{\textbf{good}}), s_4(\textit{\textbf{pretty good}})\}$. A cellphone expert is invited to conduct this assessment. It is difficult for the expert to give the evaluation information with a single LT, so it is a good choice to evaluate with LDNs. After an analysis of the iPhone X, the expert determines its performance is s_3(good) with a 30% degree of confidence, is s_4(pretty good) with 50%, and is not sure its performance at 20% because of the expert's limited knowledge. The assessment result for the iPhone X can be denoted as $LD_{\textit{\textbf{iPhoneX}}} = \{(s_3, 0.3), (s_4, 0.5)\}$, which represents incomplete information since 0.3 + 0.5 < 1.

Later, the expert evaluates the Galaxy Note 8 with LDNs. The assessment result can be represented by as $LD_{\textit{\textbf{Note}}8} = \{(s_3, 1)\}$. In this case, $LD_{\textit{\textbf{Note}}8}$ expresses complete information.

The Distance Between LDNs

The distance between two LDNs measures the degree of their divergence. There is currently no distance function suitable for measuring the distance of LDNs. Li et al. [18] proposed a novel distance function for DNs taking into consideration that the elements of FOD in DNs do not need to be mutually exclusive. Inspired by this study, we extend this novel distance function to the field of LDNs. Then, we introduce some related concepts.

Definition 7 [18] Let $S_t = \{s_0, s_1, s_2, \ldots, s_{2t}\}$ be a given LTS, where the elements of S_t are not required to be mutually exclusive. $s_i, s_j \in S_t$, the intersection area between s_i, s_j is IS_{ij}, and the union area between s_i, s_j is U_{ij}. Then, the nonexclusive degree $N-E_{ij}$ of s_i and s_j can be defined as:

$$N{-}E_{ij} = \frac{IS_{ij}}{U_{ij}} \tag{15}$$

The relative matrix R for the elements in S_t based on $N-E_{ij}$ can be established as:

$$R = \begin{bmatrix} 1 & N{-}E_{12} & \cdots & N{-}E_{1i} & \cdots & N{-}E_{12t} \\ N{-}E_{21} & 1 & \cdots & N{-}E_{2i} & \cdots & N{-}E_{22t} \\ \vdots & & 1 & & & \\ N{-}E_{i1} & N{-}E_{i2} & \cdots & 1 & \cdots & N{-}E_{i2t} \\ \vdots & & & & 1 & \\ N{-}E_{2t1} & N{-}E_{2t2} & \cdots & N{-}E_{2ti} & \cdots & 1 \end{bmatrix} \tag{16}$$

Based on the nonexclusive degree $N-E_{ij}$ and the relative matrix R, the approach for calculating the intersection degree of two sets is derived:

Definition 8 [18] Suppose $S_t = \{s_0, s_1, s_2, \ldots, s_{2t}\}$ is a given LTS, S'_1 and S'_2 are any two subsets of S_t, and the relative matrix of S'_1 and S'_2 is R. Then, the intersection degree of S'_1 and S'_2 is defined as follows:

$$I_{S'_1, S'_2} = \frac{\sum N{-}E_{ij}}{|S'_1| \times |S'_2|} \tag{17}$$

where $i \neq j$. The variable i represents the row number of each element of S'_1 in the relative matrix R; the variable j represents the column number of each element of S'_2 in R. $|S'_1|$ and $|S'_2|$ denote the cardinality of subsets S'_1 and S'_2, respectively. It must be noted that when $i = j$, $I_{S'_1, S'_2} = 1$.

Example 2 Let LTS $S_2 = \{s_0, s_1, s_2, s_3, s_4\}$ be an FOD with five elements that are not mutually exclusive. It is given that $U_{s_0,s_1} = 2$, $U_{s_1,s_2} = 3$, $U_{s_2,s_3} = 3$, $U_{s_3,s_4} = 2$, $IS_{s_0,s_1} = 0.3$, $IS_{s_1,s_2} = 0.6$, $IS_{s_2,s_3} = 0.6$, $IS_{s_3,s_4} = 0.3$, $S'_1 = \{s_1, s_2\}$, and $S'_2 = \{s_2, s_3, s_4\} \in S_2$. Then, the intersection degree of S'_1 and S'_2 can be calculated by the following steps:

Step 1: Calculate the relative matrix$R$.

Based on the given U, the ***IS*** and formula (15), we can obtain the relative matrix:

$$R=\begin{bmatrix}1 & 0.15 & 0 & 0 & 0\\ 0.15 & 1 & 0.2 & 0 & 0\\ 0 & 0.2 & 1 & 0.2 & 1\\ 0 & 0 & 0.2 & 1 & 0.15\\ 0 & 0 & 0 & 0.15 & 1\end{bmatrix}$$

Step 2: Calculate the intersection degree.

Based on the relative matrix R and formula (17), we obtain

$$I_{S_1',S_2'}=\frac{0.2+0+0+0.2}{2\times 3}=0.667.$$

To easily measure the distance between two LDNs, here, we regard an LDN as a discrete random variable, whose elements are the subsets of a given LTS with a degree of confidence distribution v. All degrees of confidence in the LDN make up with a space vector $\overrightarrow{ld}=(v_1,v_2,\cdots,v_n)^T$. Thus, the distance function between two LDNs can be defined.

Definition 9 Let $S_t=\{s_0,s_1,s_2,..s_l.,s_{2t}\}$ be a given LTS whose elements are not required to be mutually exclusive. $LD_1=\{(s_0{}^1,v_0{}^1),\cdots,(s_l{}^1,v_l{}^1),\cdots,(s_{2t}{}^1,v_{2t}{}^1)\}$ and $LD_2=\{(s_0{}^2,v_0{}^2),\cdots,(s_l{}^2,v_l{}^2),\cdots,(s_{2t}{}^2,v_{2t}{}^2)\}$ are two LDNs. Then, the distance function for LD_1 and LD_2 is

$$d(LD_1,LD_2)=\sqrt{\frac{1}{2}\left(\overrightarrow{ld_1}-\overrightarrow{ld_2}\right)^T \underset{=}{D}\,\underset{=}{I}\left(\overrightarrow{ld_1}-\overrightarrow{ld_2}\right)} \tag{18}$$

where $D_=$ and $I_=$ are two dimensional matrixes. The elements of $D_=$ are $D(A,B)=|A\cap B|/|A\cup B|(A,B\in S_t)$. The elements of $I_=$ are $I(A,B)=\sum N-E_{ij}/(|A|\cdot|B|)(A,B\in S_t,i\neq j)$ (when $i=j,I=1$), where i represents the row number that each element of A in the relative matrix R and j represents the column number of each element of B in R. $\overrightarrow{ld}_1$ and $\overrightarrow{ld}_2$ are the space vectors of LD_1 and LD_2, respectively.

Example 3 Let LTS $S_2=\{s_0,s_1,s_2,s_3,s_4\}$ be an FOD where the five elements are not mutually exclusive. It is given that $U_{s_0,s_1}=2$, $U_{s_1,s_2}=3$, $U_{s_2,s_3}=3$, $U_{s_3,s_4}=2$, $IS_{s_0,s_1}=0.3$, $IS_{s_1,s_2}=0.6$, $IS_{s_2,s_3}=0.6$, and $IS_{s_3,s_4}=0.3$. Two bodies of LDNs are given as:$LD_1=\{(s_1,0.2),(s_2,0.2),(s_3,0.6)\}$ and $LD_2=\{(s_2,0.5),(s_4,0.4)\}$. Then, the distance between LD_1 and LD_2 can be obtained by the following steps:

Step 1: Construct the space vector of LD_1 and LD_2. Here, we erase the lines having null elements in the two vectors.

$$\overrightarrow{ld_1}=\begin{pmatrix}0.2\\0.2\\0.6\\0\end{pmatrix},\overrightarrow{ld_2}=\begin{pmatrix}0\\0.5\\0\\0.4\end{pmatrix},\overrightarrow{ld_1}-\overrightarrow{ld_2}=\begin{pmatrix}0.2\\-0.3\\0.6\\-0.4\end{pmatrix}$$

Step 2: Calculate the relative matrix R.

$$R=\begin{bmatrix}1 & 0.15 & 0 & 0 & 0\\ 0.15 & 1 & 0.2 & 0 & 0\\ 0 & 0.2 & 1 & 0.2 & 0\\ 0 & 0 & 0.2 & 1 & 0.15\\ 0 & 0 & 0 & 0.15 & 1\end{bmatrix}$$

Step 3: Construct the distance matrix $D_=$. For instance, the distance between $A=\{s_1\}$ and $B=\{s_2\}$ can be calculated as $D(A,B)=|A\cap B|/|A\cup B|=0$.

$$\textit{then}\ \underset{=}{D}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Step 4: Construct the intersection matrix $I_=$ according to definition 8. For example, the intersection degree of set $A=\{s_1\}$ and $B=\{s_2\}$ can be calculated as follows:

$$|A|=1,|B|=1,I(A,B)=\frac{\sum N-E_{ij}}{|A|\cdot|B|}=\frac{0.2}{1}=0.2$$

$$\text{then}\ \underset{=}{I}=\begin{bmatrix}1&0.2&0&0\\0.2&1&0.2&0\\0&0.2&1&0.15\\0&0&0.15&1\end{bmatrix}$$

Step 5: Calculate the distance between LD_1 and LD_2.

$$d(LD_1,LD_2)=\sqrt{\frac{1}{2}\left(\overrightarrow{ld_1}-\overrightarrow{ld_2}\right)^T \underset{=}{D}\cdot\underset{=}{I}\left(\overrightarrow{ld_1}-\overrightarrow{ld_2}\right)}=0.49.$$

The Comparison Approach for LDNs

Huang [14] put forward the score function for LDNs based on the LSF, which is described as:

Definition 10 [14] Suppose there is an LDN $LD=\{(s_0,v_0),(s_1,v_1),\cdots(s_i,v_i),\cdots(s_{2t},v_{2t})\}$, where $s_i\in S$ and $\overline{f}^*$ is the LSF. Then, the score function for this LDN is:

$$I(LD)=\sum_{i=0}^{2t}\left(\overline{f}^*(s_i)v_i\right) \tag{19}$$

in which $\overline{f}^*(s_i) = i^\alpha$, $\alpha \in (0, 1]$. When $0 < \alpha < 1$, the absolute deviation between the semantic values of adjacent LTs decreases with an increase in their subscript; when $\alpha = 1$, the absolute deviation between the semantic values of adjacent LTs is constant.

However, it is not appropriate to define the LDNs' score functions using only a form of the LSFs. This is because the difference sensitivity of DEs to adjacent LTs varies with a change in the semantic environments [36], and they prefer to choose corresponding LSFs based on their preferences or cognitions. For example, if the DEs are familiar with prospect theory, they then may select the third LSF $\overline{f}_3(s_i)$ in "Linguistic Term Sets and Linguistic Scale Function" to convert the LTs, since the characteristic of function $\overline{f}_3(s_i)$ is more in line with their thinking habits. Thus, in order to adapt to different semantic environments, here, we use three kinds of LSFs in "Linguistic Term Sets and Linguistic Scale Function" to define the score function of LDNs.

Definition 11 Suppose there is an LDN $LD = \{(s_0, v_0), (s_1, v_1), \cdots(s_i, v_i), \cdots(s_{2t}, v_{2t})\}$, where $s_i \in S$ and $\overline{f}$ is the LSF. Then, the modified score function is described as:

$$I'(LD) = \sum_{i=0}^{2t} \left(\overline{f}(s_i) v_i\right) \tag{20}$$

in which $\overline{f}(s_i)$ can take the form of $f_1(s_i)$, $f_2(s_i)$ or $f_3(s_i)$, as introduced in "Linguistic Term Sets and Linguistic Scale Function".

For any two LDNs, LD_1 and LD_2, if $I'(LD_1) > I'(LD_2)$, then $LD_1 \succ LD_2$; if $I'(LD_1) = I'(LD_2)$, then we further calculate their degrees of deviation to compare them. Here, we give the degree of deviation of an LDN as follows:

Definition 12 Suppose there is an LDN $LD = \{(s_0, v_0), (s_1, v_1), \cdots(s_i, v_i), \cdots(s_{2t}, v_{2t})\}$, where $s_i \in S$ and $\overline{f}$ is the LSF, and its score value is $I'(LD)$. Then, the degree of deviation is described as:

$$\rho(LD) = \left(\sum_{i=0}^{2t} \left(\left(\overline{f}(s_i) - I'(LD)\right) v_i\right)^2\right)^{1/2} \tag{21}$$

in which $\overline{f}(s_i)$ can take the form of $f_1(s_i)$, $f_2(s_i)$, or $f_3(s_i)$ introduced in "Linguistic Term Sets and Linguistic Scale Function".

Then, the comparison approach can be defined based on the score function and the degree of deviation of the LDNs.

Definition 13 Suppose LD_1, LD_2 are two LDNs, then

(1) If $I'(LD_1) > I'(LD_2)$, then $LD_1 \succ LD_2$;
(2) If $I'(LD_1) = I'(LD_2)$, then

(i) if $\rho(LD_1) > \rho(LD_2)$, then $LD_1 \prec LD_2$;
(ii) if $\rho(LD_1) = \rho(LD_2)$, then $LD_1 = LD_2$.

Example 4 There are three LDNs, $LD_1 = \{(s_2, 0.3), (s_3, 0.5), (s_4, 0.2)\}$, $LD_2 = \{(s_3, 1)\}$, and $LD_3 = \{(s_2, 0.5), (s_4, 0.5)\}$, where the LSF $\overline{f} = f_1(s_i)$. Then, according to Eq. (20) we can calculate their score value as follows:

$$I'(LD_1) = 0.5 \times 0.3 + 0.75 \times 0.5 + 1 \times 0.2 = 0.725,$$
$$I'(LD_2) = 0.75 \times 1 = 0.75, I'(LD_3) = 0.5 \times 0.5 + 1 \times 0.5 = 0.75.$$

then, we can say LD_2 and LD_3 are greater than LD_1 because $0.75 > 0.725$.

To compare LD_2 and LD_3, we need to calculate their degrees of deviation because they have the same score values. According to Eq. (21), we have

$$\rho(LD_2) = \left((0.75 - 0.75)^2\right)^{1/2} = 0, \rho(LD_3) = \left((0.5 \times (0.5 - 0.75))^2 + (0.5 \times (1 - 0.75))^2\right)^{1/2} = 0.177.$$

then we can say LD_2 is greater than LD_3 because $0 < 0.177$. Thus, the comparison of these three LDNs is $LD_2 \succ LD_3 \succ LD_1$.

An MCDM Approach with LDNs Based on the Choquet Integral and TODIM

In this section, we present a novel cognitive MCDM approach with LDNs based on the Choquet integral and the TODIM method.

Suppose there is an MCDM problem, let $A = \{A_1, A_2, \ldots, A_m\}$ be a collection of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be the set of criteria. Let $S_t = \{s_0, s_1, s_2, .. s_l., s_{2t}\}$ be a given LTS where the LTs of S_t are not required to be mutually exclusive. The DEs use LDNs to evaluate the attribute information. $Y = [y^{ij}]_{m \times n}$ denotes the decision assessment matrix, where $y^{ij} = \bigcup_{l=0}^{2t} \{(s^{ij}{}_l, v^{ij}{}_l)\}$ represents the evaluation value of alternative A_i on attribute C_j. Then, we need to rank the alternatives by applying the proposed approach to solve this MCDM problem.

The decision steps are described as follows:

Step 1: Standardize the decision matrix.

In practical decision-making, there are two types of criteria: benefit type and cost type. To reduce the influence of different criteria types on the final decision result, the criteria need to first be translated into the same type. If the criteria are cost type, they can then be converted into a benefit type by formula (22) [14]:

$$y^{ij}=\bigcup_{l=0}^{2t}\left\{\left(s_{2t}-s^{ij}{}_{l},v^{ij}{}_{l}\right)\right\} \tag{22}$$

For convenience, the transformed decision matrix is still expressed by $Y=[y^{ij}]_{m\times n}$.

Step 2: Calculate the Shapley weights of criteria.

In real decision-making, the fuzzy density $\mu(C_j)$ of attribute C_j can be provided by DEs. Then, the $\lambda-$fuzzy measure of the criteria can be determined by formula (7), and the Shapley weight of each criterion can be calculated by formula (9).

Step 3: Calculate the distance between any two distinct alternatives' assessment values under the same criteria C_j.

$$d_j\left(y^{ij},y^{kj}\right)=\sqrt{\frac{1}{2}\left(\overrightarrow{ld}^{ij}-\overrightarrow{ld}^{kj}\right)^T\underline{\underline{D}}\,\underline{\underline{I}}\left(\overrightarrow{ld}^{ij}-\overrightarrow{ld}^{kj}\right)} \tag{23}$$

where $i\neq k$, $i,k=1,2,\cdots,m$, $j=1,2,\cdots,n$.

Step 4: Calculate the dominance values of alternative A_i over each alternative A_k on criterion C_j.

$$\varsigma_j(A_i,A_k)=\begin{cases}\sqrt{\varphi_{jr}d_j\left(y^{ij},y^{kj}\right)/\sum_{j=1}^{n}\varphi_{jr}}, & \text{if } I'\left(y^{ij}\right)-I'\left(y^{kj}\right)>0\\ 0, & \text{if } I'\left(y^{ij}\right)-I'\left(y^{kj}\right)=0\\ -\frac{1}{\theta}\sqrt{\frac{\left(\sum_{j=1}^{n}\varphi_{jr}\right)d_j\left(y^{ij}-y^{kj}\right)}{\varphi_{jr}}}, & \text{if } I'\left(y^{ij}\right)-I'\left(y^{kj}\right)<0\end{cases} \tag{24}$$

where $\varphi_{jr}=\varphi_j/\varphi_r$, and φ_j and φ_r are the Shapley weight of generic criterion C_j and the reference criterion C_r, respectively. Usually, we regard the criterion that has the maximum value of the Shapley weight as the reference criteria. $I'(y^{ij})$ and $I'(y^{kj})$ denote the score value of LDN y^{ij} and y^{kj}; $d_j(y^{ij},y^{kj})$ is the distance between the two LDNs y^{ij} and y^{kj}, and θ is the attenuation factor of the loss and denotes the loss aversion of DEs.

Step 5: Calculate the overall degree of dominance of alternative A_i over each alternative A_k.

$$\begin{aligned}o(A_i,A_k)=&\sum_{r=1}^{p}\varsigma_{(r)}(A_i,A_k)\left[\mu\left(B_{(r)}\right)-\mu\left(B_{(r-1)}\right)\right]\\&+\sum_{r=p+1}^{n}\varsigma_{(r)}(A_i,A_k)\left[\mu\left(D_{(r)}\right)-\mu\left(D_{(r+1)}\right)\right]\\&i,k=1,2,\cdots,m\end{aligned} \tag{25}$$

where $(\varsigma_{(1)}(A_i,A_k),\varsigma_{(1)}(A_i,A_k),\cdots,\varsigma_{(n)}(A_i,A_k))$ is a permutation such that $\varsigma_{(1)}(A_i,A_k)\leq\varsigma_{(2)}(A_i,A_k)\leq\cdots\leq\varsigma_{(p)}(A_i,A_k)\leq 0\leq\varsigma_{(p+1)}(A_i,A_k)\leq\cdots\leq\varsigma_{(n)}(A_i,A_k)$. $B_{(r)}=\{C_{(1)},C_{(2)},\cdots,C_{(r)}\}$ and $B_{(0)}=\phi$; $D_{(r)}=\{C_{(r)},C_{(r+1)},\cdots,C_{(n)}\}$ and $D_{(n+1)}=\phi$. $\varsigma_{(r)}(A_i,A_k)$ represents the contribution of criteria $C_{(r)}$ to function $o(A_i,A_k)$ when comparing alternatives A_i and A_k.

Step 6: Calculate the global value of the alternative A_i.

$$\delta(A_i)=\frac{\sum_{k=1}^{m}o(A_i,A_k)-min_i\left\{\sum_{k=1}^{m}o(A_i,A_k)\right\}}{max_i\left\{\sum_{k=1}^{m}o(A_i,A_k)\right\}-min_i\left\{\sum_{k=1}^{m}o(A_i,A_k)\right\}},\quad i=1,2,\cdots,m \tag{26}$$

Step 7: Rank the alternatives.

Order the alternatives by the global value $\delta(A_i)$. The best alternative has the highest global value.

An Application Example

In this section, we use the presented MCDM approach to deal with an actual MCDM problem (some of the data comes from reference [35]) to prove its feasibility. Moreover, the sensitivity analysis and comparative analysis are discussed with some existing MCDM approaches.

The Decision Problem Description

With the improvement of people's living standards, cars have become the daily means of transportation for most families. When buying a car, customers will look at it from several factors. The performance of the motor engine is an important factor because the motor engine is the core component of the car. Therefore, how to evaluate the performance of the motor engine is significant to customers and automakers.

Suppose there are four types of motor engines, A_1, A_2, A_3, A_4. Many factors influence the performance of the motor engine. The five main factors are C_1 reaction capacity, C_2 fuel consumption, C_3 sound size, C_4 vibration, and C_5 acceleration. For the above four engines, the DEs assess them with respect to the five factors. The evaluation LT $S_2=\{s_0,s_1,s_2,s_3,s_4\}$= {very poor, poor, medium, good, very good}, and the assessment information given by the DEs are denoted by LDNs. For example, the DEs gave the assessment data about the reaction capacity (C_1) of engine 3 (A_3). If the DEs think its evaluation level is "good" with a 30% degree of belief, "very good" with 60%, and 10% was not sure because of a lack of understanding this engine, then the evaluation information can be described by an LDN as $y^{31}=\{(s_3,0.3),(s_4,0.6)\}$. The DE's assessment of all engines is shown in Table 1.

The Decision-Making Steps

To select the best motor engine, the decision steps are shown in the following:

Step 1: Normalize the attribute types.

Because the criteria C_2, C_3, and C_4 are cost attributes, we need to convert the evaluation information about them to the benefit type by formula (22). The transformed decision matrix is shown in Table 2.

Step 2: Calculate the Shapley weights of criteria.

By analyzing the five attributes in the above practical problem, we find that there is an interactive among them. We then need to calculate the overall importance ($\lambda-$fuzzy measure) of each criterion.

The DEs give the fuzzy density of each criterion as follows: $\mu(C_1) = 0.3$, $\mu(C_2) = 0.35$, $\mu(C_3) = 0.2$, $\mu(C_4) = 0.15$, $\mu(C_5) = 0.25$. Then, we calculate $\lambda = -0.455$ using formula (7), and the $\lambda-$fuzzy measure of the criteria is listed in Table 3.

Then, the Shapley weight of each criterion can be obtained by formula (9):

$$\varphi_1 = 0.24, \varphi_2 = 0.28, \varphi_3 = 0.16, \varphi_4 = 0.12, \text{and } \varphi_5 = 0.20.$$

Step 3: Calculate the values of $d_j(y^{ij}, y^{kj})$.

The elements in S_2 are not mutually exclusive. Assume the union region and the intersection region between the subsets of S_2 are $U_{s_0,s_1} = 2$, $U_{s_1,s_2} = 3$, $U_{s_2,s_3} = 3$, $U_{s_3,s_4} = 2$, $IS_{s_0,s_1} = 0.3$, $IS_{s_1,s_2} = 0.6$, $IS_{s_2,s_3} = 0.6$, and $IS_{s_3,s_4} = 0.3$. Then, from formula (23), we can obtain the distance between any two distinct alternatives' evaluation values on each criterion.

$$d_1\left(y^{ij}, y^{kj}\right) = \begin{pmatrix} 0 & 0.837 & 0.602 & 1 \\ 0.837 & 0 & 0.327 & 0.906 \\ 0.602 & 0.327 & 0 & 0.867 \\ 1 & 0.906 & 0.867 & 0 \end{pmatrix}, \quad d_2\left(y^{ij}, y^{kj}\right) = \begin{pmatrix} 0 & 0.447 & 0.447 & 0.894 \\ 0.447 & 0 & 0.894 & 1 \\ 0.447 & 0.894 & 0 & 1 \\ 0.894 & 1 & 1 & 0 \end{pmatrix}$$

$$d_3\left(y^{ij}, y^{kj}\right) = \begin{pmatrix} 0 & 0.665 & 0.770 & 0.563 \\ 0.665 & 0 & 0.447 & 0.894 \\ 0.770 & 0.447 & 0 & 1 \\ 0.563 & 0.894 & 1 & 0 \end{pmatrix}, \quad d_4\left(y^{ij}, y^{kj}\right) = \begin{pmatrix} 0 & 0.461 & 0.887 & 0.887 \\ 0.461 & 0 & 1 & 1 \\ 0.887 & 1 & 0 & 0.922 \\ 0.887 & 1 & 0.922 & 0 \end{pmatrix}$$

$$d_5\left(y^{ij}, y^{kj}\right) = \begin{pmatrix} 0 & 0 & 0.584 & 1 \\ 0 & 0 & 0.584 & 1 \\ 0.584 & 0.584 & 0 & 0.801 \\ 1 & 1 & 0.801 & 0 \end{pmatrix}$$

Step 4: Calculate the dominance values $\varsigma_j(A_i, A_k)$.

In this step, we need to select the corresponding calculation formula based on the comparison of the score values. Therefore, we first calculate the score values of $y^{ij}(i = 1, 2, \cdots, m, n = 1, 2, \cdots, n)$ according to formula (20) (assume $\overline{f} = f_2(s_i)$). Then, the score values are shown in Table 4.

Next, we calculate the dominance values of alternative A_i over each alternative A_k with respect to criterion C_j using formula (24) (assume $\theta = 1$). The results are shown as follows:

$$\varsigma_1(A_i, A_k) = \begin{pmatrix} 0 & -1.87 & -1.58 & 0.49 \\ 0.45 & 0 & 0.28 & 0.47 \\ 0.38 & -1.17 & 0 & 0.46 \\ -2.04 & -1.94 & -1.90 & 0 \end{pmatrix}, \varsigma_2(A_i, A_k) = \begin{pmatrix} 0 & 0.35 & -1.26 & 0.50 \\ -1.26 & 0 & -1.79 & 0.53 \\ 0.35 & 0.50 & 0 & 0.46 \\ -1.79 & -1.89 & -1.89 & 0 \end{pmatrix}$$

$$\varsigma_3(A_i, A_k) = \begin{pmatrix} 0 & -2.03 & -2.19 & 0.30 \\ 0.33 & 0 & 0.27 & 0.38 \\ 0.35 & -1.67 & 0 & 0.40 \\ -1.88 & -2.36 & -2.5 & 0 \end{pmatrix}, \varsigma_4(A_i, A_k) = \begin{pmatrix} 0 & -1.96 & -2.72 & -2.72 \\ 0.24 & 0 & -2.89 & -2.89 \\ 0.33 & 0.35 & 0 & -2.77 \\ 0.33 & 0.35 & 0.33 & 0 \end{pmatrix} \varsigma_5(A_i, A_k) = \begin{pmatrix} 0 & 0 & 0.34 & 0.45 \\ 0 & 0 & 0.34 & 0.45 \\ 0.58 & 0.58 & 0 & 0.40 \\ -2.24 & -2.24 & -2 & 0 \end{pmatrix}.$$

Table 1 Decision matrix Y given by the DE

	C_1	C_2	C_3	C_4	C_5
A_1	$\{(s_3, 1.0)\}$	$\{(s_1, 0.5), (s_2, 0.5)\}$	$\{(s_3, 0.5), (s_4, 0.3)\}$	$\{(s_3, 0.5), (s_4, 0.5)\}$	$\{(s_3, 1.0)\}$
A_2	$\{(s_4, 1.0)\}$	$\{(s_2, 1.0)\}$	$\{(s_1, 0.5), (s_2, 0.5)\}$	$\{(s_3, 1.0)\}$	$\{(s_3, 1.0)\}$
A_3	$\{(s_3, 0.3), (s_4, 0.6)\}$	$\{(s_1, 1.0)\}$	$\{(s_2, 1.0)\}$	$\{(s_1, 1.0)\}$	$\{(s_2, 0.6), (s_3, 0.3)\}$
A_4	$\{(s_1, 1.0)\}$	$\{(s_4, 1.0)\}$	$\{(s_4, 1.0)\}$	$\{(s_0, 1.0)\}$	$\{(s_1, 1.0)\}$

Step 5: Calculate the overall degree of dominance $o(A_i, A_k)$.

Following step 4 and based on the $\lambda-$fuzzy measure values of the criteria in Table 3, the overall degree of dominance of alternative A_i over A_k can be obtained by formula (25) as follows:

$$o(A_i, A_k) = \begin{pmatrix} 0 & -1.099 & -1.470 & 0.011 \\ 0.228 & 0 & -0.815 & 0 \\ -0.123 & -0.787 & 0 & 0.009 \\ -1.775 & -1.859 & -1.824 & 0 \end{pmatrix}$$

Step 6: Calculate the global value of the alternative A_i.

The overall values of the alternatives $A_i (i = 1, 2, 3, 4)$ can be calculated according to formula (26).

$\delta(A_1) = 0.60, \delta(A_2) = 1, \delta(A_3) = 0.94$, and $\delta(A_4) = 0$.

Step 7: Rank the alternatives.

According to the global values obtained in step 6, the ranking of alternatives is $A_2 \succ A_3 \succ A_1 \succ A_4$, in which alternative A_2 is the best.

Sensitivity Analysis

In step 4 of the decision-making process, the LSF $\overline{f}$ is used to calculate the LDNs' score values, and the parameter θ is used to calculate the degree of dominance between the alternatives. Thus, in this subpart, we take different θ and $\overline{f}$ to rank the alternatives in this example to explore the impact of θ and $\overline{f}$ on the ranking results.

Case 1: The Impact of Parameter θ on the Ranking Results

In this case, we take $\overline{f} = f_2(s_i)$ and assign different values to the parameter θ to calculate the overall goals of the alternatives and rank them. The overall goals of each alternative are shown in Fig. 1, and the ranking results are listed in Table 5. From Fig. 1 we can visually see the changes in the overall goals of each alternative when θ takes different values. For alternatives A_4, its overall goals are always equal to 0. For the other alternatives, $\theta = 5.0$ is a turning point: when$0.5 \leq \theta < 5.0$, the overall goals of A_1 and A_3 both increase while the overall goals of A_2 are equal to 1; when$5.0 \leq \theta \leq 7.0$, the overall goals of A_1 and A_2 both decrease, while the overall goals of A_3 are equal to 1.

From Fig. 1 and Table 5, we find that the rankings of alternatives are always $A_2 \succ A_3 \succ A_1 \succ A_4$ when $0.5 \leq \theta < 5$, and the overall value of alternative A_2 is the largest and is always equal to 1, which means that the alternative A_2 is the optimal solution. Another point to be noted is that the overall goal of alternative A_3 becomes larger when θ is increased from 0.5 to 5. When $\theta = 5$, the overall goal of A_3 is equal to 1, which means A_3 becomes the best alternative; at the same time, the sequences of alternative A_2 is behind A_3 because its overall goal is smaller than 1.

In conclusion, the different attitudes of DEs toward risk may change their final decision result. In the TODIM approach, θ denotes the attenuation factor of losses. As analyzed in the literature [36], if the value of θ is small, the final selected best alternative is usually a lower loss one; if the value of θ is large, the final selected best alternative is usually a greater income one. Thus, comparing alternatives A_2 and A_3 in this example, the loss of alternative A_2 is lower while the income of alternative A_3 is higher. Therefore, alternative A_2 will be the best choice if the DE refuses risk; conversely, alternative A_3 will be the best choice if the DE prefers risk.

Table 2 The transformed decision matrix Y

	C_1	C_2	C_3	C_4	C_5
A_1	$\{(s_3, 1.0)\}$	$\{(s_2, 0.5), (s_3, 0.5)\}$	$\{(s_0, 0.3), (s_1, 0.5)\}$	$\{(s_0, 0.5), (s_1, 0.5)\}$	$\{(s_3, 1.0)\}$
A_2	$\{(s_4, 1.0)\}$	$\{(s_2, 1.0)\}$	$\{(s_2, 0.5), (s_3, 0.5)\}$	$\{(s_1, 1.0)\}$	$\{(s_3, 1.0)\}$
A_3	$\{(s_3, 0.3), (s_4, 0.6)\}$	$\{(s_3, 1.0)\}$	$\{(s_2, 1.0)\}$	$\{(s_3, 1.0)\}$	$\{(s_2, 0.6), (s_3, 0.3)\}$
A_4	$\{(s_1, 1.0)\}$	$\{(s_0, 1.0)\}$	$\{(s_0, 1.0)\}$	$\{(s_4, 1.0)\}$	$\{(s_1, 1.0)\}$

Table 3 λ−fuzzy measure of criteria

Criteria		Criteria		Criteria		Criteria	
$\{\phi\}$	0	$\{C_1, C_4\}$	0.43	$\{C_1, C_2, C_3\}$	0.75	$\{C_2, C_4, C_5\}$	0.67
$\{C_1\}$	0.3	$\{C_1, C_5\}$	0.52	$\{C_1, C_2, C_4\}$	0.71	$\{C_3, C_4, C_5\}$	0.55
$\{C_2\}$	0.35	$\{C_2, C_3\}$	0.52	$\{C_1, C_2, C_5\}$	0.78	$\{C_1, C_2, C_3, C_4\}$	0.85
$\{C_3\}$	0.2	$\{C_2, C_4\}$	0.48	$\{C_1, C_3, C_4\}$	0.63	$\{C_1, C_2, C_3, C_5\}$	0.91
$\{C_4\}$	0.15	$\{C_2, C_5\}$	0.56	$\{C_1, C_3, C_5\}$	0.67	$\{C_1, C_2, C_4, C_5\}$	0.88
$\{C_5\}$	0.25	$\{C_3, C_4\}$	0.34	$\{C_1, C_4, C_5\}$	0.63	$\{C_1, C_3, C_4, C_5\}$	0.81
$\{C_1, C_2\}$	0.60	$\{C_3, C_5\}$	0.43	$\{C_2, C_3, C_4\}$	0.63	$\{C_2, C_3, C_4, C_5\}$	0.81
$\{C_1, C_3\}$	0.47	$\{C_4, C_5\}$	0.38	$\{C_2, C_3, C_5\}$	0.71	$\{C_1, C_2, C_3, C_4, C_5\}$	1

Case 2: The Impact of LSFs $\overline{f}$ on the Ranking Result

In this case, we take $\theta = 1$ and assign different LSFs $\overline{f}$ to order the alternatives, with the ordering results displayed in Table 6. From Table 6, we can see that the ordering results obtained by the different LSFs are the same, i.e., although the score values of the evaluation information are changed for different LSFs (for instance, when $\overline{f} = f_1(s_i)$, the score value of y^{11} is 0.75; when $\overline{f} = f_2(s_i)$, the score value of y^{11} is 0.71), it will not affect the degree of dominance values between alternatives, so the ranking of alternatives will not be affected either. This can guarantee that the DEs can still obtain stable and reasonable ranking results when selecting different LSFs according to different semantic environments or their own preference.

Comparison Analysis

To prove the feasibility and superiority of the proposed MCDM approach, in this part, we conduct a comparative analysis that includes two cases. One is a comparison between our proposed approach and the approach developed by Han and Chen [10] with DNs based on the illustrative example in this paper. The other is a comparison between our proposed approach and the approach introduced by Huang [14] with LDNs based on the illustrative example in reference [14].

Comparison with the D-VIKOR approach by Han and Chen [10] In the MCDM approach by Han and Chen [10], the cognitive evaluation information is represented by DNs. Therefore, in order to deal with the example in this paper, we need to convert the LDNs to DNs. Here, we convert the LTs of the LDNs into numerical values of DNs by the LSFs in the condition of not changing the distribution of reliability. For example, $y^{31} = \{(s_3, 0.3), (s_4, 0.6)\}$ is an LDN, so we take $\overline{f} = f_1(s_i)$, then the transformed DN of y^{31} is $\{(0.75, 0.3), (1, 0.6)\}$. The converted information is displayed in Table 7. Then, we use the extended D-VIKOR approach to rank the alternatives. The ranking results are outlined in Table 8.

Table 4 Score values of y^{ij}

	C_1	C_2	C_3	C_4	C_5
A_1	0.71	0.605	0.145	0.145	0.71
A_2	1	0.5	0.605	0.29	0.71
A_3	0.81	0.71	0.5	0.71	0.51
A_4	0.29	0	0	1	0.29

From Table 8, we can see that the ranking result obtained by the D-VIKOR approach in [10] coincides with those by our proposed approach (when $\theta \geq 5.0$). The ranking result is $A_3 \succ A_2 \succ A_1 \succ A_4$, in which the optimal alternative is A_3. However, there appears to be an inconsistency between the ranking results obtained by the two approaches when $0.5 \leq \theta < 5.0$. The inconsistency is reflected in the sorting of alternative A_2 and A_3, and it may be caused by the difference in the basic ideas contained in the two approaches. The approach by Han and Chen [10] assumes that the DEs are completely rational in the decision-making process, yet our proposed approach is based on "bounded rationality," which considers the DEs' psychological behavior as a reference dependence and cognition toward loss. From the analysis in "The Decision-Making Steps", it can be seen that the psychological behavior of DEs indeed influences the decision results (if the DE hates risk, he/she may choose A_2; if the DE prefers risk, he/she may choose A_3). Thus, compared with the approach in [10], our proposed approach is more reasonable to actual decision-making problems according to the risk attitude of DEs. In addition, the approach by Han and Chen [10] uses DNs to express evaluation information, whereas the proposed approach uses LDNs, which inherits the advantages of LTs and DNs. The LDNs can not only express uncertain and

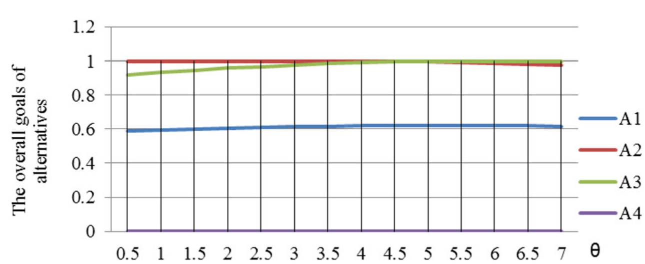

Fig. 1 Overall goals of each alternative

Table 5 Ranking results obtained by utilizing different θ

θ	Overall goals of alternative A_i	Ranking
$\theta = 0.5$	$\delta(A_1) = 0.589$, $\delta(A_2) = 1$, $\delta(A_3) = 0.921$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\theta = 1.0$	$\delta(A_1) = 0.595$, $\delta(A_2) = 1$, $\delta(A_3) = 0.936$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\theta = 1.5$	$\delta(A_1) = 0.601$, $\delta(A_2) = 1$, $\delta(A_3) = 0.948$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\theta = 2.0$	$\delta(A_1) = 0.605$, $\delta(A_2) = 1$, $\delta(A_3) = 0.959$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\theta = 2.5$	$\delta(A_1) = 0.609$, $\delta(A_2) = 1$, $\delta(A_3) = 0.969$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\theta = 3.0$	$\delta(A_1) = 0.613$, $\delta(A_2) = 1$, $\delta(A_3) = 0.978$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\theta = 3.5$	$\delta(A_1) = 0.616$, $\delta(A_2) = 1$, $\delta(A_3) = 0.985$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\theta = 4.0$	$\delta(A_1) = 0.619$, $\delta(A_2) = 1$, $\delta(A_3) = 0.992$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\theta = 4.5$	$\delta(A_1) = 0.621$, $\delta(A_2) = 1$, $\delta(A_3) = 0.999$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\theta = 5.0$	$\delta(A_1) = 0.621$, $\delta(A_2) = 0.996$, $\delta(A_3) = 1$, $\delta(A_4) = 0$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$\theta = 5.5$	$\delta(A_1) = 0.620$, $\delta(A_2) = 0.991$, $\delta(A_3) = 1$, $\delta(A_4) = 0$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$\theta = 6.0$	$\delta(A_1) = 0.619$, $\delta(A_2) = 0.986$, $\delta(A_3) = 1$, $\delta(A_4) = 0$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$\theta = 6.5$	$\delta(A_1) = 0.618$, $\delta(A_2) = 0.982$, $\delta(A_3) = 1$, $\delta(A_4) = 0$	$A_3 \succ A_2 \succ A_1 \succ A_4$
$\theta = 7.0$	$\delta(A_1) = 0.617$, $\delta(A_2) = 0.978$, $\delta(A_3) = 1$, $\delta(A_4) = 0$	$A_3 \succ A_2 \succ A_1 \succ A_4$

incomplete information such as the DNs but can also express complex information by the LTs. Therefore, our proposed approach provides DEs a more accurate way to give evaluation information by LDNs.

Comparison with the approach based on the linguistic D numbers prioritized weighted average (LD-PWA) operator [14] In this part, we use the proposed approach to solve the investment decision-making problem in reference [14]. Here, we consider two situations: the first is that the attributes are independent, the second is that the attributes are interactive.

In the first situation, the fuzzy measures of the criteria are equivalent to the general weights and their sum must equal to 1. Therefore, we take the average weights of the attributes in [14] as their fuzzy densities such that $\mu(C_1) = 0.46$, $\mu(C_2) = 0.3$, $\mu(C_3) = 0.16$ and $\mu(C_4) = 0.08$. In the second situation, the fuzzy measures not only represent the weight on the criteria but also the combination of criteria and then their sum does not equal 1. Since the precedence of criteria is $C_1 \succ C_2 \succ C_3 \succ C_4$ in this example, we assume their fuzzy densities are $\mu(C_1) = 0.4$, $\mu(C_2) = 0.37$, $\mu(C_3) = 0.25$, $\mu(C_4) = 0.2$. Then, the weights of the criteria can be calculated such that $\omega(C_1) = 0.32$, $\omega(C_2) = 0.32$, $\omega(C_3) = 0.21$, $\omega(C_4) = 0.15$. The ranking results of the alternatives by the proposed approach in these two situations and the approach developed in [14] are shown in Table 9.

Table 6 Ranking results obtained by utilizing different $\overline{f}$

$\overline{f}$	Overall goals of alternative A_i	Ranking
$\overline{f} = f_1(s_i)$	$\delta(A_1) = 0.595$, $\delta(A_2) = 1$, $\delta(A_3) = 0.935$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\overline{f} = f_2(s_i)$	$\delta(A_1) = 0.595$, $\delta(A_2) = 1$, $\delta(A_3) = 0.935$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\overline{f} = f_3(s_i)$	$\delta(A_1) = 0.595$, $\delta(A_2) = 1$, $\delta(A_3) = 0.935$, $\delta(A_4) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4$

As shown in Table 9, the ranking results by the proposed approach in situation 1 are the same as those of Huang's approach [14], proving that the proposed approach is effective because these two approaches do not consider the interaction among attributes in situation 1. However, there is a difference between the ranking results obtained by the proposed approach in situation 2 and the approach in [14]. The difference is reflected in the sorting of alternatives A_4 and A_5. This difference may be because Huang's approach only considers the priority among criteria without considering the interaction among them. Unlike Huang's approach, the proposed approach can effectively deal with the correlation information among the criteria by the Choquet integral operator. Additionally, the weights of criteria calculated in approach [14] can only measure the importance of the criteria but cannot reflect the complementary or redundancy interaction among them, with the Shapley weights in our proposed approach making up for the insufficiency. Therefore, our proposed approach can not only handle LDNs MCDM problems where the criteria are independent, but it can also handle those with interactive criteria.

The comparisons for the three MCDM approaches are listed in Table 10.

From Table 10, we can summarize the major preponderances of the presented MCDM approach as follows:

(1) The proposed MCDM approach expresses the cognitive information by the LDNs. The LDNs can express incomplete and uncertain cognitive information more flexibly and effectively since they integrate the merits of LTs and DNs. Then it is more convenient for DEs to evaluate the alternatives by the form of the LDNs.

Table 7 The decision matrix denoted by DNs

	C_1	C_2	C_3	C_4	C_5
A_1	{(0.75, 1.0)}	{(0.5, 0.5), (0.75, 0.5)}	{(0, 0.3), (0.25, 0.5)}	{(0, 0.5), (0.25, 0.5)}	{(0.75, 1.0)}
A_2	{(1, 1.0)}	{(0.5, 1.0)}	{(0.5, 0.5), (0.75, 0.5)}	{(0.25, 1.0)}	{(0.75, 1.0)}
A_3	{(0.75, 0.3), (1, 0.6)}	{(0.75, 1.0)}	{(0.5, 1.0)}	{(0.75, 1.0)}	{(0.5, 0.6), (0.75, 0.3)}
A_4	{(0.25, 1.0)}	{(0, 1.0)}	{(0, 1.0)}	{(1, 1.0)}	{(0.25, 1.0)}

Table 8 Ranking results by the proposed approach and the D-VIKOR approach in [10]

Approaches	Values	Ranking
The proposed approach ($0.5 \le \theta < 5.0$)	See Table 5	$A_2 \succ A_3 \succ A_1 \succ A_4$
The proposed approach ($\theta \ge 5.0$)	See Table 5	$A_3 \succ A_2 \succ A_1 \succ A_4$
The D-VIKOR approach [10]	$Q(A_1) = 0.518$, $\delta(A_2) = 0.374$, $\delta(A_3) = 0$, $\delta(A_4) = 1$	$A_3 \succ A_2 \succ A_1 \succ A_4$

(2) The proposed MCDM approach extends the traditional TODIM approach to handle MCDM problems with LDNs that enlarges the application range of the TODIM approach. Moreover, compared with the MCDM approaches that deal with LDNs presented in the literature [14, 33], which assume that DEs are completely rational in their decisions, the proposed approach considers the bounded rationality of DEs. Thus, the decision results produced by the proposed approach are much closer to the actual situations.

(3) The proposed MCDM approach inherits the characteristics of the TODIM approach and the Choquet integral. It can not only take the influence of the limited cognitive abilities of DEs on the decision-making results into account, but it can also deal with the correlation between cognitive preferences.

Conclusion

LDNs inherit the merits of LTs and DNs in that they can not only express incomplete and hesitant cognitive information such as DNs, but they can also express complex information flexibly such as LTs. Thus, it is valuable to study MCDM problems under the LDNs decision environment. In this paper, we introduced the LDNs to represent the cognitive information in MCDM problems and reviewed the comparison method between two LDNs. Then, we proposed the distance function and modified the score function for LDNs. Later, considering the DEs' bounded rationality behaviors when addressing practical MCDM problems and the phenomenon that criteria or cognitive preferences are not independent, we developed a novel cognitive MCDM approach by extending the TODIM method and the Choquet integral to handle the cognitive

Table 9 Ranking results by the proposed approach and the approach in [14]

Approaches	Values	Ranking
The proposed approach (case 1) and ($\theta = 1.0$)	$\delta(A_1) = 0.66$, $\delta(A_2) = 1$, $\delta(A_3) = 0.82$, $\delta(A_4) = 0$, $\delta(A_5) = 0.09$	$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$
The proposed approach (case 2) and ($\theta = 1.0$)	$\delta(A_1) = 0.82$, $\delta(A_2) = 1$, $\delta(A_3) = 0.83$, $\delta(A_4) = 0.04$, $\delta(A_5) = 0$	$A_2 \succ A_3 \succ A_1 \succ A_4 \succ A_5$
The approach based on LD-PWA operator [14]	$I(A_1) = 4.16$, $\delta(A_2) = 4.51$, $\delta(A_3) = 4.43$, $\delta(A_4) = 3.90$, $\delta(A_5) = 4.1$	$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4$

Table 10 The comparisons for three MCDM approaches

Approaches	Information type	Whether it considers the interactivity of attributes	Whether it considers the bounded rationality of DEs'
The D-VIKOR approach [10]	DNs	NO	NO
The approach based on the LD-PWA operator [14]	LDNs (combine the advantages of DNs and LTs)	NO	NO
The proposed approach	LDNs (combine the advantages of DNs and LTs)	YES (Choquet integral)	YES (TODIM)

information expressed by LDNs. Finally, we verified the effectiveness and superiority of the proposed approach by solving the practical problem of selecting motor engines and then compared it with other existing MCDM approaches [10, 14]. The main contributions of this study are that (1) LDNs were introduced to express uncertain and hesitant cognitive information in MCDM problems, (2) a new distance measure and a comparison approach of LDNs were proposed' and (3) the proposed approach not only considered the cognition toward loss of DEs but also processed the relation among criteria or cognitive preferences.

In future studies, we will focus on utilizing the proposed MCDM approach to solve more complex MCDM problems with different information. For example, the proposed approach can be extended to address such MCDM problems as the transformer condition assessment [29], supplier selection [5], and product engineering [18]. Additionally, further studies can extend the TODIM and Choquet integral to other fuzzy environments, such as the picture fuzzy linguistic information [22], the interval-valued fuzzy soft sets (IVFSs) [26], and the IFSs [11, 12].

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Compliance with Ethical Standards

Conflicts of Interest The authors declare that they have no conflict of interest.

Research Involving Human Participants and/or Animals This article does not contain any studies with human participants or animals performed by any of the authors.

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