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Improvements on Correlation Coefficients of Hesitant Fuzzy Sets and Their Applications

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Abstract

Hesitant fuzzy set (HFS) can express the hesitancy and uncertainty according to human's cognitions and knowledge. The decision making with HFSs can be regarded as a cognitive computation process. Decision making based on information measures is a hot topic, among which correlation coefficient is an important direction. Although many correlation coefficients of HFSs have been proposed in the previous papers, they suffer from different counter-intuitions to a certain extent. Therefore, we mainly focus on improving these counter-intuitions of the existing correlation coefficients of HFSs in this paper. We point out the counter-intuitions of the existing correlation coefficients of HFSs and analyze the reasons of them in the view of the rigorous mathematics and stochastic process rules. We improve these counter-intuitions and develop the correct versions. Moreover, we use two examples about medical diagnosis and cluster analysis to compare the improved correlation coefficients with the existing ones. The improved correlation coefficients can handle the examples well. Further, combining with the comparison analysis, the accuracy and discrimination property of the improved correlation coefficients are demonstrated in detail, which shows the advantages of them. The notion of the improved correlation coefficients can benefit other types of fuzzy sets too.

Keywords Correlation coefficients · Hesitant fuzzy sets (HFSs) · Weighted correlation coefficients · Decision making · Clustering

Introduction

Cognitive computation aims to handle the lifeless or dull issues in view of human brain idea or human sense-making. It becomes more and more popular nowadays and promotes machines to have reasoning abilities which are analogous to human with the rapid development of artificial intelligence (AI). The core of this process is to simulate the human's cognition and make the decision. As to the decision-making process, it is more efficient for decision makers to provide their cognitions of the alternatives by means of a fuzzy set [1–3]. Among the fuzzy sets, the hesitant fuzzy set (HFS) is just such a set to express the hesitancy and uncertainty of the expert's cognitions more comprehensively than other extensions of fuzzy sets when making a decision.

Since Torra [4] introduced the HFS, it has been paid great attention to, which makes a profound influence on both theory and applications. To the best of our knowledge, the existing analyses of HFS in decision making focus on three aspects: (1) the information measures including the distance and similarity [5–8], entropy [9, 10], gray relational analysis [11, 12], and correlation measures; (2) the aggregation operators, such as the basic aggregation operators [13, 14], Frank aggregation operators [15, 16], Heronian mean and Muirhead mean aggregation operators [17, 18], and power and power Bonferroni means aggregation operators [19–21]; (3) the consensus models which pay attention to the additive consistency [22, 23], multiplicative consistency [24, 25], and consistency measures [26–28]. Among these three hot topics, the correlation measure is a significant direction under the information measures. Therefore, in this paper, we mainly focus on this point and aim at improving the existing correlation measures for HFS.

Correlation measures how well two variables move together in a linear relation, which is one of the most widely used indices in various applications. As to the correlation measures for HFS, different types of correlations and correlation coefficients have been proposed. It should be noticed that the existing correlation

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measures for HFS can be mostly divided into two categories: in the information energy view and in the statistics view. Xu and Xia [29] firstly defined the basic correlation coefficients for hesitant fuzzy set (HFE) in the information energy view. Later, Chen et al. [30] extended these correlation coefficients from HFEs to HFSs and interval-valued HFSs (IVHFSs). Liao et al. [31, 32] extended this notion to the correlations and correlation coefficients of hesitant fuzzy linguistic term sets (HFLTSs). Moreover, Wang et al. [33], Ye [34], and Tyagi [35] et al. extended these information energy correlation coefficients to the dual hesitant fuzzy sets (DHFSs). Afterwards, Farhadinia [36] extended them into correlation coefficients for interval-valued DHFSs (IVDHFSs). Meng et al. [37, 38] improved these information energy correlation coefficients of HFSs and IVHFSs based on the Shapley function and fuzzy measure, which relaxes the length and order of the memberships. Besides, Liao et al. [39] proposed some novel correlation coefficients between HFSs in the statistics view, which also do not need to consider the length and order of the memberships. Yang et al. [40] extended Liao et al.'s notion to the hesitant multiplicative set (HMS) directly. Guan et al. [41] and Sun et al. [42] improved these statistics correlation coefficients with synthetic correlation coefficients by taking into the variance and hesitancy degree characters. Recently, Sahin et al. [43], Das et al. [44], and Ji et al. [45] investigated the correlation measures of single-valued neutrosophic hesitant fuzzy sets, hesitant fuzzy soft sets, and multi-hesitant fuzzy linguistic term sets (MHFLTSs), respectively, but these correlation measures are still based on the information energy correlation coefficients. Dong et al. [46] proposed a cosine similarity of HFLTSs, which is actually a correlation coefficient in the information energy view. In addition, the correlation coefficients of the HFSs have been applied into such fields as decision making [34–44], clustering analysis [30, 33, 39–41], pattern recognition [38, 41], medical diagnosis [36, 39, 41], linguistic computing [31, 32, 45, 46], and feature selection [47–49].

Although, there are so many correlation coefficients for HFSs, all of them are derived from the information energy correlation coefficients in [29, 30] and the statistics correlation coefficients in [39]. However, these three basic correlation coefficients suffer from different drawbacks. The correlation coefficients in [29] are only in the HFEs domain. The energy and correlation of HFSs in [30] are counter-intuitive especially when comparing with the weighted styles. It disobeys the theorem that if the weight vector is the same for each x_i, $i = 1, 2, \cdots, n$ in X, then the weighted correlation turns into the ordinary correlation. Furthermore, Liao et al. in [39] defined the novel mean, variance, correlation, and correlation coefficient of HFSs and extended them to the weighted types. However, these weighted types are unreasonable, which lie not only in the repeated multiple factor $\frac{1}{n}$, but also are controversial in the form of these definitions themselves.

Therefore, the purpose of this study is to improve these counter-intuitions of the above correlation coefficients. Firstly, we extend Xu and Xia's correlation coefficients in [29] from HFEs to HFSs domain by adding the weight of them. Thereafter, we show that the definition 6 and definition 7 in Chen et al.'s paper [32] do not obey the mathematics and stochastic process rules. We improve them by a more rigorous derivation process which is in mathematics view. In addition, we improve the counter-intuition of the weighted mean, variance, correlation, and correlation coefficient of the HFSs in Liao et al.'s study [39] and provide the corrected representations.

The remainder of this paper is organized as follows. In the "Preliminaries" section, we briefly review some basic concepts of HFSs and the present correlation coefficients between HFSs. In the "Counter-Intuitive Analysis" section, we debate the counter-intuition of these correlation coefficients in detail. Some improved representations of these correlation coefficients between HFEs and HFSs are proposed in the "Some Improved Versions" section. In the "Suggested Compared Example" section, the improved correlation coefficients are validated in detail by using two comparative numerical examples. The conclusions and future challenges of this study are summarized in the "Conclusions" section.

Preliminaries

In this section, we review the HFSs and the correlation coefficients in [29, 30, 39].

Hesitant Fuzzy Sets

A decision-making scenario is a standard cognitive computation process. When an expert makes a decision, he or she may hesitate to choose the exact membership degree in [0, 1]. Instead, HFS is just such a cognitive set which is good at dealing with the situations that people have disagreements or hesitancy when deciding something.

Suppose that $X = \{x_1, x_2, \cdots, x_n\}$ is a reference set, a hesitant fuzzy set (HFS) A on X is defined in terms of a function $h_A(x)$ when applied to X which returns a subset of [0, 1],

$$A = \{\langle x, h_A(x)\rangle | x \in X\} \tag{1}$$

where $h_A(x)$ is a set of some different values in [0, 1], representing the possible membership degrees of the element $x \in X$ to the set A. For convenience, Xia and Xu [13] call $h_A(x)$ a hesitant fuzzy element (HFE), which is a basic unit of HFS.

The Present Correlation Coefficients in [29]

Xu and Xia firstly proposed five types of correlation coefficients for HFEs in [29].

For two HFEs $h_A(x_i) = \left\{\gamma_{Ai1}, \gamma_{Ai2}, \cdots, \gamma_{Ail_{Ai}}\right\}$ and $h_B(x_i) = \left\{\gamma_{Bi1}, \gamma_{Bi2}, \cdots, \gamma_{Bil_{Bi}}\right\}$ on a fixed set $X = \{x_1, x_2, \cdots, x_n\}$, l_{Ai}

and l_{Bi} are the number of the membership values in $h_A(x_i)$ and $h_B(x_i)$, respectively, assuming that $l_{Ai} = l_{Bi}$, then the correlation coefficient between the HFE $h_A(x_i)$ and $h_B(x_i)$ is defined as

$$\rho_1(h_A, h_B) = \frac{\sum_{j=1}^{l_{Ai}} \gamma_{Aij}^{\sigma(j)} \cdot \gamma_{Bij}^{\sigma(j)}}{\left(\sum_{j=1}^{l_{Ai}} \left(\gamma_{Aij}^{\sigma(j)}\right)^2 \cdot \sum_{j=1}^{l_{Bi}} \left(\gamma_{Bij}^{\sigma(j)}\right)^2\right)^{1/2}} \tag{2}$$

$$\rho_2(h_A, h_B) = \frac{\sum_{j=1}^{l_{Ai}} \gamma_{Aij}^{\sigma(j)} \cdot \gamma_{Bij}^{\sigma(j)}}{\max\left\{\sum_{j=1}^{l_{Ai}} \left(\gamma_{Aij}^{\sigma(j)}\right)^2, \sum_{j=1}^{l_{Bi}} \left(\gamma_{Bij}^{\sigma(j)}\right)^2\right\}} \tag{3}$$

$$\rho_3(h_A, h_B) = \frac{\sum_{j=1}^{l_{Ai}} \left(\gamma_{Aij}^{\sigma(j)} - \overline{h}_A(x_i)\right) \cdot \left(\gamma_{Bij}^{\sigma(j)} - \overline{h}_B(x_i)\right)}{\left(\sum_{j=1}^{l_{Ai}} \left(\gamma_{Aij}^{\sigma(j)} - \overline{h}_A(x_i)\right)^2 \cdot \sum_{j=1}^{l_{Bi}} \left(\gamma_{Bij}^{\sigma(j)} - \overline{h}_B(x_i)\right)^2\right)^{1/2}} \tag{4}$$

$$\rho_4(h_A, h_B) = \frac{\sum_{j=1}^{l_{Ai}} \left(\gamma_{Aij}^{\sigma(j)} - \overline{h}_A(x_i)\right) \cdot \left(\gamma_{Bij}^{\sigma(j)} - \overline{h}_B(x_i)\right)}{\max\left\{\sum_{j=1}^{l_{Ai}} \left(\gamma_{Aij}^{\sigma(j)} - \overline{h}_A(x_i)\right)^2, \sum_{j=1}^{l_{Bi}} \left(\gamma_{Bij}^{\sigma(j)} - \overline{h}_B(x_i)\right)^2\right\}} \tag{5}$$

$$\rho_5(h_A, h_B) = \sum_{j=1}^{l_{Ai}} \frac{\min_j |\gamma_{Aij}^{\sigma(j)} - \gamma_{Bij}^{\sigma(j)}| + \max_j |\gamma_{Aij}^{\sigma(j)} - \gamma_{Bij}^{\sigma(j)}|}{|\gamma_{Aij}^{\sigma(j)} - \gamma_{Bij}^{\sigma(j)}| + \max_j |\gamma_{Aij}^{\sigma(j)} - \gamma_{Bij}^{\sigma(j)}|} \tag{6}$$

where $\gamma_{Aij}^{\sigma(j)}$ and $\gamma_{Bij}^{\sigma(j)}$ are the jth membership values in $h_A(x_i)$ and $h_B(x_i)$, which satisfy $\gamma_{Aij}^{\sigma(j)} \geq \gamma_{Aij}^{\sigma(j+1)}$ and $\gamma_{Bij}^{\sigma(j)} \geq \gamma_{Bij}^{\sigma(j+1)}$, $\overline{h}_A(x_i)$, and $\overline{h}_B(x_i)$ are the mean of the HFE$h_A(x_i)$ and $h_B(x_i)$, respectively,

$$\overline{h}_A(x_i) = \frac{1}{l_{Ai}} \sum_{j=1}^{l_{Ai}} \gamma_{Aij}, \overline{h}_B(x_i) = \frac{1}{l_{Bi}} \sum_{j=1}^{l_{Bi}} \gamma_{Aij}, \quad i = 1, 2, \cdots, n \tag{7}$$

These above five correlation coefficients are only limit to the HFEs domain, thus Chen et al. [30] extended them into the HFSs domain.

The Present Correlation Coefficients in [30]

In Chen et al.'s paper [30], they defined the correlation coefficients for HFSs based on the definition of the hesitant fuzzy energy and correlation.

For two HFSs $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in X, i = 1, 2, \cdots, n\}$ and $B = \{\langle x_i, h_B(x_i)\rangle | x_i \in X, i = 1, 2, \cdots, n\}$ on the fixed set $X = \{x_1, x_2, \cdots, x_n\}$ with $h_A(x_i) = \{\gamma_{Ai1}, \gamma_{Ai2}, \cdots, \gamma_{Ail_{Ai}}\}$ and $h_B(x_i) = \{\gamma_{Bi1}, \gamma_{Bi2}, \cdots, \gamma_{Bil_{Bi}}\}$, l_{Ai} and l_{Bi} are the numbers of the membership values in $h_A(x_i)$ and $h_B(x_i)$, respectively, assuming that $l_{Ai} = l_{Bi}$.

The informational energy of the HFS $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in X, i = 1, 2, \cdots, n\}$ is defined as

$$E(A) = \sum_{i=1}^{n} \left(\frac{1}{l_{Ai}} \sum_{j=1}^{l_{Ai}} \left(\gamma_{Aij}^{\sigma(j)}\right)^2\right) \tag{8}$$

For two HFSs $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in X, i = 1, 2, \cdots, n\}$ and $B = \{\langle x_i, h_B(x_i)\rangle | x_i \in X, i = 1, 2, \cdots, n\}$, their correlation is defined by

$$C(A, B) = \sum_{i=1}^{n} \left(\frac{1}{l_{Ai}} \sum_{j=1}^{l_{Ai}} \gamma_{Aij}^{\sigma(j)} \cdot \gamma_{Bij}^{\sigma(j)}\right) \tag{9}$$

Thus, based on the informational energy and the correlation, two kinds of correlation coefficients between two HFSs A and B are defined as follows:

$$\rho_6(A, B) = \frac{C(A, B)}{[C(A, A)]^{\frac{1}{2}} \cdot [C(B, B)]^{\frac{1}{2}}} = \frac{\sum_{i=1}^{n} \left(\frac{1}{l_{Ai}} \sum_{j=1}^{l_{Ai}} \gamma_{Aij}^{\sigma(j)} \cdot \gamma_{Bij}^{\sigma(j)}\right)}{\left[\sum_{i=1}^{n} \left(\frac{1}{l_{Ai}} \sum_{j=1}^{l_{Ai}} \left(\gamma_{Aij}^{\sigma(j)}\right)^2\right)\right]^{\frac{1}{2}} \cdot \left[\sum_{i=1}^{n} \left(\frac{1}{l_{Bi}} \sum_{j=1}^{l_{Bi}} \left(\gamma_{Bij}^{\sigma(j)}\right)^2\right)\right]^{\frac{1}{2}}} \tag{10}$$

$$\rho_7(A, B) = \frac{C(A, B)}{\max\{C(A, A), C(B, B)\}} = \frac{\sum_{i=1}^{n} \left(\frac{1}{l_{Ai}} \sum_{j=1}^{l_{Ai}} \gamma_{Aij}^{\sigma(j)} \cdot \gamma_{Bij}^{\sigma(j)}\right)}{\max\left\{\sum_{i=1}^{n} \left(\frac{1}{l_{Ai}} \sum_{j=1}^{l_{Ai}} \left(\gamma_{Aij}^{\sigma(j)}\right)^2\right), \sum_{i=1}^{n} \left(\frac{1}{l_{Bi}} \sum_{j=1}^{l_{Bi}} \left(\gamma_{Bij}^{\sigma(j)}\right)^2\right)\right\}} \tag{11}$$

If we take the weight into consideration, let the weight vector of X be $\mathbf{w} = \{w_1, w_2, \cdots, w_n\}$, $\sum_{i=1}^{n} w_i = 1$, and $i = 1, 2, \cdots, n$, the correlation is extended into the weighted correlation as

$$C_w(A, B) = \sum_{i=1}^{n} w_i \cdot \left(\frac{1}{l_{Ai}} \sum_{j=1}^{l_{Ai}} \gamma_{Aij}^{\sigma(j)} \cdot \gamma_{Bij}^{\sigma(j)}\right) \tag{12}$$

Then, the weighted correlation coefficients between two HFSs A and B are given by

$$\rho_{w6}(A, B) = \frac{C_w(A, B)}{[C_w(A, A)]^{\frac{1}{2}} \cdot [C_w(B, B)]^{\frac{1}{2}}} = \frac{\sum_{i=1}^{n} w_i \cdot \left(\frac{1}{l_{Ai}} \sum_{j=1}^{l_{Ai}} \gamma_{Aij}^{\sigma(j)} \cdot \gamma_{Bij}^{\sigma(j)}\right)}{\left[\sum_{i=1}^{n} w_i \cdot \left(\frac{1}{l_{Ai}} \sum_{j=1}^{l_{Ai}} \left(\gamma_{Aij}^{\sigma(j)}\right)^2\right)\right]^{\frac{1}{2}} \cdot \left[\sum_{i=1}^{n} w_i \cdot \left(\frac{1}{l_{Bi}} \sum_{j=1}^{l_{Bi}} \left(\gamma_{Bij}^{\sigma(j)}\right)^2\right)\right]^{\frac{1}{2}}} \tag{13}$$

$$\rho_{w7}(A,B)=\frac{C_w(A,B)}{\max\{C_w(A,A),C_w(B,B)\}}=\frac{\sum_{i=1}^{n}w_i\cdot\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\gamma_{Aij}^{\sigma(j)}\cdot\gamma_{Bij}^{\sigma(j)}\right)}{\max\left\{\sum_{i=1}^{n}w_i\cdot\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}\right)^2\right),\sum_{i=1}^{n}w_i\cdot\left(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}\right)^2\right)\right\}} \tag{14}$$

Although the correlation coefficients proposed by Chen et al. [30] is in information energy view, it lies in the interval [0, 1], which is not satisfied with the mathematics and stochastic process rules that the correlation coefficient should be in [− 1, 1].

The Present Correlation Coefficients in [39]

In Liao et al.'s paper [39], they defined the mean, the variance, and the correlation of the HFSs.

The mean of a HFS $A=\{\langle x_i,h_A(x_i)\rangle|x_i\in X,i=1,2,\cdots,n\}$ with $h_A(x_i)=\{\gamma_{Ai1},\gamma_{Ai2},\cdots,\gamma_{Ail_{Ai}}\},i=1,2,\cdots,n$ is defined as

$$\overline{A}=\frac{1}{n}\sum_{i=1}^{n}\overline{h}_A(x_i)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\gamma_{Aij}\right) \tag{15}$$

where $\overline{h}_A(x_i)$ is the mean of the HFE $h_A(x_i)$

$$\overline{h}_A(x_i)=\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\gamma_{Aij},i=1,2,\cdots,n \tag{16}$$

Based on the mean of the HFS, the variance of a HFS A is defined as

$$Var(A)=\frac{1}{n}\sum_{i=1}^{n}\left[\overline{h}_A(x_i)-\overline{A}\right] \tag{17}$$

Then, for two HFSs $A=\{\langle x_i,h_A(x_i)\rangle|x_i\in X,i=1,2,\cdots,n\}$ and $B=\{\langle x_i,h_B(x_i)\rangle|x_i\in X,i=1,2,\cdots,n\}$, the correlation between them is defined as

$$C_L(A,B)=\frac{1}{n}\sum_{i=1}^{n}\left[\overline{h}_A(x_i)-\overline{A}\right]\cdot\left[\overline{h}_B(x_i)-\overline{B}\right]=\frac{1}{n}\sum_{i=1}^{n}\left[\overline{h}_A(x_i)-\frac{1}{n}\sum_{i=1}^{n}\overline{h}_A(x_i)\right]\cdot\left[\overline{h}_B(x_i)-\frac{1}{n}\sum_{i=1}^{n}\overline{h}_B(x_i)\right] \tag{18}$$

where $\overline{h}_B(x_i)$ is the mean of the HFE $h_B(x_i)$.

$$\overline{h}_B(x_i)=\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\gamma_{Bij},i=1,2,\cdots,n \tag{19}$$

Thus, based on which, the correlation coefficient between HFSs A and B is defined as

$$\rho_8(A,B)=\frac{C_L(A,B)}{[C_L(A,A)]^{\frac{1}{2}}\cdot[C_L(B,B)]^{\frac{1}{2}}}=\frac{\sum_{i=1}^{n}\left[\overline{h}_A(x_i)-\frac{1}{n}\sum_{i=1}^{n}\overline{h}_A(x_i)\right]\cdot\left[\overline{h}_B(x_i)-\frac{1}{n}\sum_{i=1}^{n}\overline{h}_B(x_i)\right]}{\left\{\sum_{i=1}^{n}\left[\overline{h}_A(x_i)-\frac{1}{n}\sum_{i=1}^{n}\overline{h}_A(x_i)\right]^2\cdot\sum_{i=1}^{n}\left[\overline{h}_B(x_i)-\frac{1}{n}\sum_{i=1}^{n}\overline{h}_B(x_i)\right]^2\right\}^{\frac{1}{2}}} \tag{20}$$

If we take the weight into consideration, let the weight vector of X be $\mathbf{w}=\{w_1,w_2,\cdots,w_n\}$, $\sum_{i=1}^{n}w_i=1$, and $i=1,2,\cdots,n$. Liao et al. extended the mean, the variance, and the correlation into the weighted mean, variance, and correlation of the HFSs, respectively.

The weighted mean of the HFS A is defined as

$$\overline{A}_w=\frac{1}{n}\sum_{i=1}^{n}w_i\cdot\overline{h}_A(x_i)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{w_i}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\gamma_{Aij}\right) \tag{21}$$

The weighted variance of the HFS A is defined as

$$\mathrm{Var}_w(A)=\frac{1}{n}\sum_{i=1}^{n}\left[w_i\cdot\overline{h}_A(x_i)-\overline{A}_w\right] \tag{22}$$

The weighted correlation between HFSs A and B is defined as

$$C_{wL}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\left[w_i\cdot\overline{h}_A(x_i)-\overline{A}_w\right]\cdot\left[w_i\cdot\overline{h}_B(x_i)-\overline{B}_w\right]=\frac{1}{n}\sum_{i=1}^{n}\left[w_i\cdot\overline{h}_A(x_i)-\frac{1}{n}\sum_{i=1}^{n}w_i\cdot\overline{h}_A(x_i)\right]\cdot\left[w_i\cdot\overline{h}_B(x_i)-\frac{1}{n}\sum_{i=1}^{n}w_i\cdot\overline{h}_B(x_i)\right] \tag{23}$$

Then, the weighted correlation coefficient between two HFSs A and B is given by

$$\rho_{w8}(A,B)=\frac{C_{wL}(A,B)}{[C_{wL}(A,A)]^{\frac{1}{2}}\cdot[C_{wL}(B,B)]^{\frac{1}{2}}}=\frac{\sum_{i=1}^{n}\left[w_i\cdot\overline{h}_A(x_i)-\frac{1}{n}\sum_{i=1}^{n}w_i\cdot\overline{h}_A(x_i)\right]\cdot\left[w_i\cdot\overline{h}_B(x_i)-\frac{1}{n}w_i\cdot\sum_{i=1}^{n}\overline{h}_B(x_i)\right]}{\left\{\sum_{i=1}^{n}\left[w_i\cdot\overline{h}_A(x_i)-\frac{1}{n}\sum_{i=1}^{n}w_i\cdot\overline{h}_A(x_i)\right]^2\cdot\sum_{i=1}^{n}\left[w_i\cdot\overline{h}_B(x_i)-\frac{1}{n}\sum_{i=1}^{n}w_i\cdot\overline{h}_B(x_i)\right]^2\right\}^{\frac{1}{2}}} \tag{24}$$

Counter-Intuitive Analysis

In this section, we will analyze the counter-intuition of the definitions in Chen et al.'s paper [32] and Liao et al.'s study [34].

Counter-Intuition of the Correlation in [30]

We have pointed out that the weighted correlation between two HFSs A and B in [30] is counter-intuitive. The weighted correlation they defined disobeys the theorem that if the weight vector is the same for each x_i, $i = 1, 2, \cdots, n$ in X, that is, the weight vector be $\mathbf{w} = \left\{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right\}$, then the weighted correlation turns into the ordinary correlation. Hold this in mind, we let the weight vector be $\mathbf{w} = \left\{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right\}$, then the weighted correlation does not turn to the correlation as Eq. (9), instead, it turns to the following expression:

$$C_r(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\gamma_{Aij}^{\sigma(j)}\cdot\gamma_{Bij}^{\sigma(j)}\right) \tag{25}$$

The difference between Chen et al.'s correlation (9) and the corrected one (25) is the multiple factor $\frac{1}{n}$. Therefore, in order to make the correlation and weighted type unified, the representation of the correlation of the HFS $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in X, i = 1, 2, \cdots, n\}$ should be corrected as the Eq. (25) in our paper.

Followed by the same idea of the correlation, the informational energy of the HFS$A = \{\langle x_i, h_A(x_i)\rangle | x_i \in X, i = 1, 2, \cdots, n\}$ should be corrected as

$$E_r(A) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}\right)^2\right) \tag{26}$$

Similarly, the correlation coefficients between two HFSs A and B should also be corrected as

$$\begin{aligned}\rho_{r6}(A,B) &= \frac{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\gamma_{Aij}^{\sigma(j)}\cdot\gamma_{Bij}^{\sigma(j)}\right)}{\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}\right)^2\right)\right]^{\frac{1}{2}}\cdot\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}\right)^2\right)\right]^{\frac{1}{2}}} \\ &= \frac{\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\gamma_{Aij}^{\sigma(j)}\cdot\gamma_{Bij}^{\sigma(j)}\right)}{\left[\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}\right)^2\right)\right]^{\frac{1}{2}}\cdot\left[\sum_{i=1}^{n}\left(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}\right)^2\right)\right]^{\frac{1}{2}}}\end{aligned} \tag{27}$$

$$\begin{aligned}\rho_{r7}(A,B) &= \frac{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\gamma_{Aij}^{\sigma(j)}\cdot\gamma_{Bij}^{\sigma(j)}\right)}{\max\left\{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}\right)^2\right), \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}\right)^2\right)\right\}} \\ &= \frac{\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\gamma_{Aij}^{\sigma(j)}\cdot\gamma_{Bij}^{\sigma(j)}\right)}{\max\left\{\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}\right)^2\right), \sum_{i=1}^{n}\left(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}\right)^2\right)\right\}}\end{aligned} \tag{28}$$

Remark 1 It is noticed that the corrected correlation coefficients are the same with the correlation coefficients defined in Chen et al.'s paper [29]. It is because that the multiple factor $\frac{1}{n}$ is reduced when divided. However, the multiple factor $\frac{1}{n}$ should not be neglected in the calculation process.

Remark 2 The same counter-intuitive representation should also be corrected in the definitions of the correlation and correlation coefficients between the IVHFSs in Chen et al.'s paper [29].

Suggested Modified Example 1 Following the example in [29], we modify the calculation process of the correlation coefficients between HFSs.

Let two HFSs A and B in $X = \{x_1, x_2, \cdots, x_n\}$ are

$$A = \{\langle x_1, \{0.7, 0.5\}\rangle, \langle x_2, \{0.9, 0.8, 0.6\}\rangle, \langle x_3, \{0.5, 0.4, 0.2\}\rangle\},$$

$$B = \{\langle x_1, \{0.4, 0.2\}\rangle, \langle x_2, \{0.8, 0.5, 0.4\}\rangle, \langle x_3, \{0.7, 0.6, 0.3\}\rangle\}.$$

By using Eq. (25), we calculate the corrected correlation as

$$\begin{aligned}C_r(A,B) &= \frac{1}{3}\cdot\sum_{i=1}^{3}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\gamma_{Aij}^{\sigma(j)}\cdot\gamma_{Bij}^{\sigma(j)}\right) \\ &= \frac{1}{3}\cdot\left(\frac{1}{2}\sum_{j=1}^{2}\gamma_{A1j}^{\sigma(j)}\cdot\gamma_{B1j}^{\sigma(j)} + \frac{1}{3}\sum_{j=1}^{3}\gamma_{A2j}^{\sigma(j)}\cdot\gamma_{B2j}^{\sigma(j)} + \frac{1}{3}\sum_{j=1}^{3}\gamma_{A3j}^{\sigma(j)}\cdot\gamma_{B3j}^{\sigma(j)}\right) \\ &= \frac{1}{3}\times\left\{\frac{1}{2}\times(0.7\times0.4 + 0.5\times0.2)\right. \\ &\quad + \frac{1}{3}\times(0.9\times0.8 + 0.8\times0.5 + 0.6\times0.4) \\ &\quad \left. + \frac{1}{3}\times(0.5\times0.7 + 0.4\times0.6 + 0.2\times0.3)\right\} \\ &= 0.287.\end{aligned}$$

Based on the Eq. (26), we obtain the corrected informational energy

$$\begin{aligned}C_{r_1}(A,A) &= \frac{1}{3}\cdot\sum_{i=1}^{3}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}\right)^2\right) \\ &= \frac{1}{3}\cdot\left(\frac{1}{2}\sum_{j=1}^{2}\left(\gamma_{A1j}^{\sigma(j)}\right)^2 + \frac{1}{3}\sum_{j=1}^{3}\left(\gamma_{A2j}^{\sigma(j)}\right)^2 + \frac{1}{3}\sum_{j=1}^{3}\left(\gamma_{A3j}^{\sigma(j)}\right)^2\right) \\ &= \frac{1}{3}\times\left\{\frac{1}{2}\times\left(0.7^2 + 0.5^2\right) + \frac{1}{3}\times\left(0.9^2 + 0.8^2 + 0.6^2\right)\right. \\ &\quad \left. + \frac{1}{3}\times\left(0.5^2 + 0.4^2 + 0.2^2\right)\right\} \\ &= 0.374\end{aligned}$$

Similarly,

$$\begin{aligned}C_{r_1}(B,B) &= \frac{1}{3}\cdot\sum_{i=1}^{3}\left(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}\right)^2\right) \\ &= \frac{1}{3}\cdot\left(\frac{1}{2}\sum_{j=1}^{2}\left(\gamma_{B1j}^{\sigma(j)}\right)^2 + \frac{1}{3}\sum_{j=1}^{3}\left(\gamma_{B2j}^{\sigma(j)}\right)^2 + \frac{1}{3}\sum_{j=1}^{3}\left(\gamma_{B3j}^{\sigma(j)}\right)^2\right) \\ &= \frac{1}{3}\times\left\{\frac{1}{2}\times\left(0.4^2 + 0.2^2\right) + \frac{1}{3}\times\left(0.8^2 + 0.5^2 + 0.4^2\right)\right. \\ &\quad \left. + \frac{1}{3}\times\left(0.7^2 + 0.6^2 + 0.3^2\right)\right\} \\ &= 0.254\end{aligned}$$

Finally, the correlation coefficient is calculated as

$$\rho_{r6}(A,B) = \frac{C_{r_1}(A,B)}{[C_{r_1}(A,A)]^{\frac{1}{2}}\cdot[C_r(B,B)]^{\frac{1}{2}}} = \frac{0.287}{\sqrt{0.374}\cdot\sqrt{0.254}} = 0.9311$$

This result is the same with Chen et al.'s paper which is explained in Remark 1.

Counter-Intuition of the Correlations in [39]

Corrigendum 1 In the Liao et al.'s paper [39], the definition of the variance of a HFS A omits the square denotation and should be corrected as follows

$$\mathrm{Var}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\overline{h}_A(x_i) - \overline{A}\right]^2 \tag{29}$$

Corrigendum 2 Similarly, the weighted variance of a HFS A should also be corrected as

$$\mathrm{Var}_w(A) = \frac{1}{n}\sum_{i=1}^{n}\left[w_i\cdot\overline{h}_A(x_i) - \overline{A}_w\right]^2 \tag{30}$$

Theorem 1 It is common that if the weight vector is the same for each x_i, $i = 1, 2, \cdots, n$ in X, that is, the weight vector $\mathbf{w} = \left\{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right\}$, then the weighted mean turns into the ordinary mean.

Holding this Theorem 1 in mind, we let the weight vector in Liao et al.'s study [39] be $\mathbf{w} = \left\{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right\}$, then the weighted mean dose not turn into the same equation as the Definition 4 in [39]. Thus, it is counter-intuitive; the reason is that the multiple factor $\frac{1}{n}$ is repeated. Therefore, in order to overcome this shortcoming, the representation of weighted mean of a HFS $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in X, i = 1, 2, \cdots, n\}$ should be corrected as

$$\overline{A}_{rw} = \sum_{i=1}^{n} w_i\cdot\overline{h}_A(x_i) = \sum_{i=1}^{n} w_i\cdot\left(\frac{1}{l_{Ai}}\sum_{k=1}^{l_{Ai}}\gamma_{Aik}\right) \tag{31}$$

Similarly, the problem that the repeated multiple factor $\frac{1}{n}$ also exists in the definition of the weighted variance, correlation, and correlation coefficient of the HFS in [39]. So, if we get rid of the repeated multiple factor $\frac{1}{n}$, then the weighted variance, correlation, and correlation coefficient of the HFS in Liao et al.'s study turn into the following equations:

The weighted variance is defined as

$$\mathrm{Var}_w(A) = \sum_{i=1}^{n}\left[w_i\cdot\overline{h}_A(x_i) - \overline{A}_{rw}\right]^2 \tag{32}$$

The weighted correlation is defined as

$$C_w(A,B) = \sum_{i=1}^{n}\left[w_i\cdot\overline{h}_A(x_i) - \overline{A}_{rw}\right]\cdot\left[w_i\cdot\overline{h}_B(x_i) - \overline{B}_{rw}\right] = \sum_{i=1}^{n}\left[w_i\cdot\overline{h}_A(x_i) - \sum_{i=1}^{n} w_i\cdot\overline{h}_A(x_i)\right]\cdot\left[w_i\cdot\overline{h}_B(x_i) - \sum_{i=1}^{n} w_i\cdot\overline{h}_B(x_i)\right] \tag{33}$$

The weighted correlation coefficient is defined as

$$\rho_w(A,B) = \frac{C_w(A,B)}{[C_w(A,A)]^{\frac{1}{2}}\cdot[C_w(B,B)]^{\frac{1}{2}}} = \frac{\sum_{i=1}^{n}\left[w_i\cdot\overline{h}_A(x_i) - \sum_{i=1}^{n} w_i\cdot\overline{h}_A(x_i)\right]\cdot\left[w_i\cdot\overline{h}_B(x_i) - w_i\cdot\sum_{i=1}^{n}\overline{h}_B(x_i)\right]}{\left\{\sum_{i=1}^{n}\left[w_i\cdot\overline{h}_A(x_i) - \sum_{i=1}^{n} w_i\cdot\overline{h}_A(x_i)\right]^2\cdot\sum_{i=1}^{n}\left[w_i\cdot\overline{h}_B(x_i) - \sum_{i=1}^{n} w_i\cdot\overline{h}_B(x_i)\right]^2\right\}^{\frac{1}{2}}} \tag{34}$$

Unfortunately, we find that these three above equations are not satisfied with the idea of Theorem 1. That is to say, if we let the weight vector in these three above equations be $\mathbf{w} = \left\{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right\}$, the weighted variance, correlation, and correlation coefficient of the HFS do not turn into the same ordinary variance, correlation, and correlation coefficient of the HFS in Liao et al.'s study. It indicates that the problems in these three definitions not only lie in the repeated multiple factor $\frac{1}{n}$, but also in the definition themselves.

Some Improved Versions

In this section, we will improve the correlation definitions in [29, 30, 39].

The Improvement of Xu and Xia's Correlation Coefficients

Actually, if we let the weight vector of X be $\mathbf{w} = \{w_1, w_2, \cdots, w_n\}$, $\sum_{i=1}^{n} w_i = 1$, and $i = 1, 2, \cdots, n$, we can extend the Xu and Xia's correlation coefficients between HFEs into the correlation coefficients between HFSs.

For two HFSs $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in X, i = 1, 2, \cdots, n\}$, $B = \{\langle x_i, h_B(x_i)\rangle | x_i \in X, i = 1, 2, \cdots, n\}$ on the fixed set $X = \{x_1, x_2, \cdots, x_n\}$ with $h_A(x_i) = \{\gamma_{Ai1}, \gamma_{Ai2}, \cdots, \gamma_{Ail_{Ai}}\}$, $h_B(x_i) = \{\gamma_{Bi1}, \gamma_{Bi2}, \cdots, \gamma_{Bil_{Bi}}\}$, $i = 1, 2, \cdots, n$, l_{Ai}, and l_{Bi} are the numbers of the membership values in $h_A(x_i)$ and $h_B(x_i)$ respectively, assuming that $l_{Ai} = l_{Bi}$, then the correlation coefficients between two HFSs A and B can be represented as

$$\rho_1(A,B)=\sum_{i=1}^{n} w_i\cdot\frac{\sum_{j=1}^{l_{Ai}}\gamma_{Aij}^{\sigma(j)}\cdot\gamma_{Bij}^{\sigma(j)}}{\left(\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}\right)^2\cdot\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}\right)^2\right)^{1/2}} \tag{35}$$

$$\rho_2(A,B)=\sum_{i=1}^{n} w_i\cdot\frac{\sum_{j=1}^{l_{Ai}}\gamma_{Aij}^{\sigma(j)}\cdot\gamma_{Bij}^{\sigma(j)}}{\max\left\{\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}\right)^2,\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}\right)^2\right\}} \tag{36}$$

$$\rho_3(A,B)=\sum_{i=1}^{n} w_i\cdot\frac{\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)\cdot\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)}{\left(\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)^2\cdot\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)^2\right)^{1/2}} \tag{37}$$

$$\rho_4(A,B)=\sum_{i=1}^{n} w_i\cdot\frac{\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)\cdot\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)}{\max\left\{\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)^2,\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)^2\right\}} \tag{38}$$

$$\rho_5(A,B)=\sum_{i=1}^{n} w_i\cdot\sum_{j=1}^{l_{Ai}}\frac{\min_j|\gamma_{Aij}^{\sigma(j)}-\gamma_{Bij}^{\sigma(j)}|+\max_j|\gamma_{Aij}^{\sigma(j)}-\gamma_{Bij}^{\sigma(j)}|}{|\gamma_{Aij}^{\sigma(j)}-\gamma_{Bij}^{\sigma(j)}|+\max_j|\gamma_{Aij}^{\sigma(j)}-\gamma_{Bij}^{\sigma(j)}|} \tag{39}$$

Remark 3 When comparing these new definitions with the correlation coefficients by Chen et al. [30]. We can see that these five definitions are from different view. The definitions by Chen et al. calculate the weighted HFS correlation firstly and then give the correlation coefficients of the HFS. However, these five definitions calculate the correlation coefficients of the HFE firstly and then weight them to construct the correlation coefficients of the HFS. If the weight vector be $\mathbf{w}=\left\{\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right\}$, then they are equal in value.

The Improvement of Chen et al.'s Correlation Coefficients

Chen et al.'s correlation coefficients are not satisfied with the mathematics and stochastic process rules in value, so we improve them to lie in [− 1, 1] in the following:

The improved correlation of the HFSs A is denoted as

$$C_{\text{imp}}=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)\cdot\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)\right) \tag{40}$$

Then, two improved correlations between the HFSs A and B are defined as

$$\rho_{\text{imp6}}(A,B)=\frac{C_{\text{imp}}(A,B)}{\left[C_{\text{imp}}(A,A)\right]^{\frac{1}{2}}\cdot\left[C_{\text{imp}}(B,B)\right]^{\frac{1}{2}}}=\frac{\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)\cdot\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)\right)}{\left[\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)^2\right)\right]^{\frac{1}{2}}\cdot\left[\sum_{i=1}^{n}\left(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)^2\right)\right]^{\frac{1}{2}}} \tag{41}$$

$$\rho_{\text{imp7}}(A,B)=\frac{C_{\text{imp}}(A,B)}{\left[C_{\text{imp}}(A,A)\right]^{\frac{1}{2}}\cdot\left[C_{\text{imp}}(B,B)\right]^{\frac{1}{2}}}=\frac{\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)\cdot\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)\right)}{\max\left\{\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)^2\right),\sum_{i=1}^{n}\left(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)^2\right)\right\}} \tag{42}$$

$$\overline{h}_A(x_i)=\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\gamma_{Aij},\ \overline{h}_B(x_i)=\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\gamma_{Aij},\quad i=1,2,\cdots,n \tag{43}$$

Note that the factor $\frac{1}{n}$ is reduced.

The improved weighted correlations between the HFSs A and B are defined as

$$\rho_{\text{impw6}}(A,B)=\frac{C_{\text{impw}}(A,B)}{\left[C_{\text{impw}}(A,A)\right]^{\frac{1}{2}}\cdot\left[C_{\text{impw}}(B,B)\right]^{\frac{1}{2}}}=\frac{w_i\cdot\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)\cdot\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)\right)}{\left[w_i\cdot\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)^2\right)\right]^{\frac{1}{2}}\cdot\left[w_i\cdot\sum_{i=1}^{n}\left(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)^2\right)\right]^{\frac{1}{2}}} \tag{44}$$

$$\rho_{\text{impw7}}(A,B)=\frac{C_{\text{impw}}(A,B)}{\max\left\{\left[C_{\text{impw}}(A,A)\right],\left[C_{\text{impw}}(B,B)\right]\right\}}=\frac{w_i\cdot\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)\cdot\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)\right)}{\max\left\{w_i\cdot\sum_{i=1}^{n}\left(\frac{1}{l_{Ai}}\sum_{j=1}^{l_{Ai}}\left(\gamma_{Aij}^{\sigma(j)}-\overline{h}_A(x_i)\right)^2\right),w_i\cdot\sum_{i=1}^{n}\left(\frac{1}{l_{Bi}}\sum_{j=1}^{l_{Bi}}\left(\gamma_{Bij}^{\sigma(j)}-\overline{h}_B(x_i)\right)^2\right)\right\}} \tag{45}$$

The Improvement of Liao et al.'s Correlation Coefficients

We improve the definitions of the weighted variance, correlation, and correlation coefficient of the HFS in Liao et al.'s study, which are shown in the following:

Table 1 Symptom characteristics for the considered diagnoses in terms of HFSs

Diagnoses	Temperature	Headache	Cough	Stomach pain	Chester pain
Viral fever	{0.6,0.4,0.3}	{0.7,0.5,0.3,0.2}	{0.5,0.3}	{0.5,0.4,0.3,0.2,0.1}	{0.5,0.4,0.2,0.1}
Malaria	{0.9,0.8,0.7}	{0.5,0.3,0.2,0.1}	{0.2,0.1}	{0.6,0.5,0.3,0.2,0.1}	{0.4,0.3,0.2,0.1}
Typhoid	{0.6,0.3,0.1}	{0.9,0.8,0.7,0.6}	{0.5,0.3}	{0.5,0.4,0.3,0.2,0.1}	{0.6,0.4,0.3,0.1}
Stomach problem	{0.5,0.4,0.2}	{0.4,0.3,0.2,0.1}	{0.4,0.3}	{0.9,0.8,0.7,0.6,0.5}	{0.5,0.4,0.2,0.1}
Chest problem	{0.3,0.2,0.1}	{0.5,0.3,0.2,0.1}	{0.3,0.2}	{0.7,0.6,0.5,0.3,0.2}	{0.9,0.8,0.7,0.6}

The corrected weighted variance is defined as

$$\mathrm{Var}_{rw}(A) = w_i \cdot \sum_{i=1}^{n} \left[\overline{h}_A(x_i) - \overline{A}_{rw}\right]^2 = w_i \cdot \sum_{i=1}^{n} \left[\overline{h}_A(x_i) - \sum_{i=1}^{n} w_i \cdot \overline{h}_A(x_i)\right]^2 \quad (46)$$

The corrected weighted correlation is defined as

$$C_{rw}(A,B) = \sum_{i=1}^{n} w_i \cdot \left[\overline{h}_A(x_i) - \overline{A}_{rw}\right] \cdot \left[\overline{h}_B(x_i) - \overline{B}_{rw}\right] = w_i \cdot \sum_{i=1}^{n} \left[\overline{h}_A(x_i) - \sum_{i=1}^{n} w_i \cdot \overline{h}_A(x_i)\right] \cdot \left[\overline{h}_B(x_i) - \sum_{i=1}^{n} w_i \cdot \overline{h}_B(x_i)\right] \quad (47)$$

The corrected weighted correlation coefficient is defined as

$$\rho_{rw8}(A,B) = \frac{C_{rw}(A,B)}{[C_{rw}(A,A)]^{\frac{1}{2}} \cdot [C_{rw}(B,B)]^{\frac{1}{2}}} = \frac{w_i \cdot \sum_{i=1}^{n} \left[\overline{h}_A(x_i) - \sum_{i=1}^{n} w_i \cdot \overline{h}_A(x_i)\right] \cdot \left[\overline{h}_B(x_i) - \sum_{i=1}^{n} w_i \cdot \overline{h}_B(x_i)\right]}{\left\{ w_i \cdot \sum_{i=1}^{n} \left[\overline{h}_A(x_i) - \sum_{i=1}^{n} w_i \cdot \overline{h}_A(x_i)\right]^2 \cdot w_i \cdot \sum_{i=1}^{n} \left[\overline{h}_B(x_i) - \sum_{i=1}^{n} w_i \cdot \overline{h}_B(x_i)\right]^2 \right\}^{\frac{1}{2}}} \quad (48)$$

Apparently, if we let the weight vector in these three above equations be $\mathbf{w} = \left\{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right\}$, the corrected weighted variance, correlation, and correlation coefficient of the HFS turn into the same ordinary variance, correlation, and correlation coefficient of the HFS in Liao et al.'s study. It proves that our corrected representations are more appropriate in the mathematics expression.

We can also give the definition of the weighted correlation coefficient of the HFS as follows

$$\rho_{rw9}(A,B) = \frac{C_{rw}(A,B)}{\max\{[C_{rw}(A,A)] \cdot [C_{rw}(B,B)]\}} = \frac{w_i \cdot \sum_{i=1}^{n} \left[\overline{h}_A(x_i) - \sum_{i=1}^{n} w_i \cdot \overline{h}_A(x_i)\right] \cdot \left[\overline{h}_B(x_i) - \sum_{i=1}^{n} w_i \cdot \overline{h}_B(x_i)\right]}{\max\left\{ w_i \cdot \sum_{i=1}^{n} \left[\overline{h}_A(x_i) - \sum_{i=1}^{n} w_i \cdot \overline{h}_A(x_i)\right]^2, w_i \cdot \sum_{i=1}^{n} \left[\overline{h}_B(x_i) - \sum_{i=1}^{n} w_i \cdot \overline{h}_B(x_i)\right]^2 \right\}} \quad (49)$$

Suggested Compared Example

In this section, we use two examples about medical diagnosis and cluster analysis to compare the improved weighted correlation coefficient with the existing correlation coefficients.

Apply the Improved Correlation Coefficient between HFSs to Medical Diagnosis

Example 2 [39] Suppose that a doctor wants to make a proper diagnosis D = {viral fever, malaria, typhoid, stomach problem, chest problem} for a set of patients P = {Al, Bob, Joe, Ted} with the values of symptoms V = {temperature, headache, cough, stomach pain, chest pain}. As in many cases, we cannot get the definite measuring instruments, so it is impossible to get the crisp values of the symptoms. Instead, these values are

Table 2 Symptom characteristics for the considered patients in terms of HFSs

Patients	Temperature	Headache	Cough	Stomach pain	Chester pain
Al	{0.9,0.7,0.5}	{0.4,0.3,0.2,0.1}	{0.4,0.3}	{0.6,0.5,0.4,0.2,0.1}	{0.4,0.3,0.2,0.1}
Bob	{0.5,0.4,0.2}	{0.5,0.4,0.3,0.1}	{0.2,0.1}	{0.9,0.8,0.6,0.5,0.4}	{0.5,0.4,0.3,0.2}
Joe	{0.9,0.7,0.6}	{0.7,0.4,0.3,0.1}	{0.3,0.2}	{0.6,0.4,0.3,0.2,0.1}	{0.6,0.3,0.2,0.1}
Ted	{0.8,0.7,0.5}	{0.6,0.5,0.4,0.2}	{0.4,0.3}	{0.6,0.4,0.3,0.2,0.1}	{0.5,0.4,0.2,0.1}

Table 3 Correlation coefficient values by the improved Chen et al.'s weighted correlation coefficient (44)

Patients	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem	Diagnosis result
Al	0.9134	0.9163	0.9664	*0.9684*	0.9215	Cannot be diagnosed
Bob	0.9289	0.9568	0.9034	*0.9696*	0.9515	Cannot be diagnosed
Joe	0.9627	*0.9674*	0.8696	0.9215	0.9652	Cannot be diagnosed
Ted	0.9360	0.9468	0.9102	*0.9839*	0.9442	*Stomach problem*

usually described in terms of HFEs. The medical symptom characteristics of the considered diagnoses are shown in Table 1. The symptoms of the patients are given in Table 2.

To make a diagnosis for each patient, we will calculate the correlation coefficients between the symptoms' characteristic of each diagnosis and that of each patient. We use the improved Chen et al.'s and Liao et al.'s weighted correlation coefficients (Eqs.. (44, 45, and 48)) to compare with each other. Suppose that the weight vector of symptoms V be $\mathbf{w} = \{0.3, 0.3, 0.2, 0.1, 0.1\}$, then the weighted correlation coefficient values can be obtained, which are shown in Tables 3, 4, 5 and 6.

From Table 3 and Table 4, we can see that the correlation coefficients of the diagnosis are so close that we are not confident to make the decision, so we regard them as the ineffective results in this example and we do not discuss them again.

From Table 5, it is clear to see that Al, Joe, and Ted suffer from malaria, and Bob suffers from stomach problem. While Table 6 implies that Al, Bob, and Joe suffer from malaria, it cannot decide which diagnosis Ted suffers from. It is because that the correlation coefficient of viral fever and the malaria is 0.9254 and 0.9362, they are so close that it is not easy to decide which one Ted suffers from.

Comparing the correlation coefficient results in Table 5 with those in Table 6, some interesting findings can be also obtained. Firstly, we get two different diagnosis results by using the improved Liao et al.'s weighted correlation coefficient and Liao et al.'s weighted correlation coefficients. Let us look into the symptom characteristics of malaria, stomach problem, and those of Bob. It is obvious that under the given weight vector, Bob's symptoms is more correlated to those of stomach problem than malaria. However, Liao et al.'s weighted correlation coefficient formula gets the wrong result because of its counter-intuitive definition. While the improved Liao et al.'s weighted correlation coefficient overcomes this shortcoming, which is more reasonable in accuracy.

In addition, the discrimination ability of the improved Liao et al.'s weighted correlation coefficient is better than Liao et al.'s weighted correlation coefficient. To show a better understanding, we compare a new figure of the correlation coefficient values by using both weighted correlation coefficient approach within the same domain as shown in Fig. 1. From Fig. 1, the solid lines express the correlation coefficient values by using Liao et al.'s weighted correlation coefficient, while the dashed lines express the correlation coefficient values by using the improved weighted correlation coefficient. Through comparison, it is very hard for us to distinguish the diagnoses with the solid lines clearly, because the correlation coefficient values is so close that we cannot distinguish the diagnoses. Especially for the patient Ted, the correlation coefficient of the two diagnoses, viral fever and the malaria, is 0.9254 and 0.9362. However, we can distinguish the diagnoses clearly through the dashed lines, which show that the improved Liao et al.'s weighted correlation coefficient is better than Liao et al.'s weighted correlation coefficient in discrimination.

Apply the Correlation Coefficient Between HFSs to Cluster Analysis

In this section, we apply the improved correlation coefficients to a cluster scenario and compare them with Chen et al.'s and Liao et al.'s methods. The purpose of this paper is to improve

Table 4 Correlation coefficient values by the improved Chen et al.'s weighted correlation coefficient (45)

Patients	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem	diagnosis result
Al	0.7974	0.8256	0.8176	*0.8619*	0.8303	*Stomach problem*
Bob	0.7983	0.8757	0.7525	*0.8765*	0.8708	Cannot be diagnosed
Joe	*0.8955*	0.7078	0.8347	0.6660	0.7062	*Malaria*
Ted	0.8231	0.8469	0.7757	*0.8692*	0.8446	*Stomach problem*

Table 5 Correlation coefficient values by using the improved Liao et al.'s weighted correlation coefficient (48)

Patients	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem	diagnosis result
Al	0.3987	*0.9355*	− 0.6135	0.2214	− 0.4840	*Malaria*
Bob	− 0.4587	0.3398	− 0.2061	*0.6897*	0.3124	*Stomach problem*
Joe	0.5195	*0.9883*	− 0.3083	− 0.0044	− 0.4666	*Malaria*
Ted	0.6752	*0.9447*	− 0.2117	− 0.1138	− 0.6202	*Malaria*

Table 6 Correlation coefficient values by using the Liao et al.'s weighted correlation coefficient

Patients	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem	Diagnosis result
Al	0.7671	*0.9743*	0.2703	0.8756	− 0.0568	*Malaria*
Bob	0.7525	*0.8157*	0.6024	0.7715	0.2711	*Malaria*
Joe	0.8461	*0.9819*	0.4532	0.8441	0.1643	*Malaria*
Ted	0.9245	0.9362	0.5743	0.8516	0.2046	Cannot be diagnosed

the existing correlation coefficients, so we do not intend to propose a new clustering algorithm but use the algorithm in [29, 39] to compare them. The cluster analysis algorithm is described as below.

Cluster Analysis Algorithm

Step 1. Let $A_i (i = 1, 2, \cdots, m)$ be a set of HFSs in $X = \{x_1, x_2, \cdots, x_n\}$. Calculate the correlation coefficients $\rho_{ij} = \rho(A_i, A_j)$ between these HFSs by the improved correlation coefficients and construct a correlation matrix $C = (\rho_{ij})_{m\times m}$.

Step 2. Check whether $C = (\rho_{ij})_{m\times m}$ is an equivalent correlation coefficient matrix by the examination rule: if it satisfies $C^2 \subseteq C$, where $C^2 = C\circ C = \left(\rho'_{ij}\right)_{m\times m}, \left(\rho'_{ij}\right)_{m\times m} = \max\limits_k\left\{\min\left\{\rho_{ik}, \rho_{kj}\right\}\right\}, i, j = 1, 2, \cdots, m,$ $C = (\rho_{ij})_{m\times m}$ is the equivalent correlation coefficient matrix. Else, construct the equivalent correlation coefficient matrix $C^{2^k}: C \to C^2 \to \cdots \to C^{2^k} \to \cdots$, until $C^{2^k} = C^{2^{k+1}}$.

Step 3. For a given confidence level $\lambda \in [0, 1]$, we construct a λ-cutting matrix $C_\lambda = ({}_\lambda\rho_{ij})_{m\times m}$ to classify the HFSs by the classify principle, where

$$ {}_\lambda\rho_{ij} = \begin{cases} 0 & if\, \rho_{ij} < \lambda, \\ 1 & if\, \rho_{ij} \geq \lambda, \end{cases} \quad i, j = 1, 2, \cdots, m. \tag{50}$$

Classify principle: If all elements of the ith line (column) in C_λ are the same as the corresponding elements of the jth line (column) in C_λ, then the HFSs A_i and A_j are of the same type. By means of this principle, we can classify all these HFSs.

Example 3 [29, 39] The risk evaluation organizations aim to evaluate 10 firms $A_i (i = 1, 2, \cdots, 10)$ on 5 criteria (x_1 managers' work experience, x_2 profitability, x_3 operating capacity, x_4 debt-paying ability, and x_5 market competition, whose weight vector is $w = (0.15, 0.3, 0.2, 0.25, 0.1)^T$) and classify them according to their risk of failure, given that these organizations will have different experiences and levels of knowledge, cognition, and so on. It is obviously a cognitive computation process, so in order to make a better assessment, these evaluation data are represented by HFSs, which are shown in Table 7.

According to the procedure of the cluster analysis algorithm, we use the improved Chen et al.'s and Liao et al.'s correlation coefficients (Eqs. (44, 45, 48, 49)) to classify these 10 firms. The calculation process is in the following.

Step 1. Calculate the correlation coefficients between these 10 firms by the improved correlation coefficients and construct the correlation coefficient matrixes as follows.

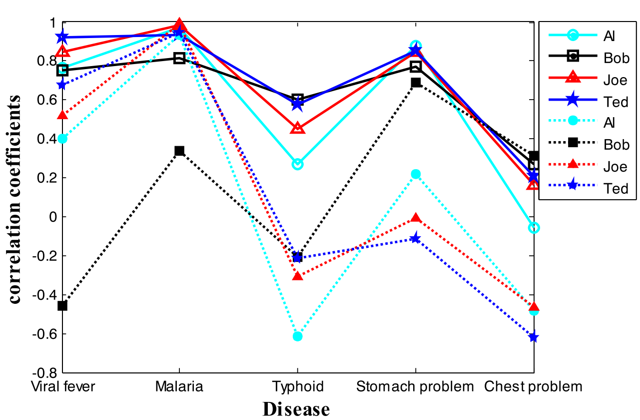

Fig. 1 Correlation coefficient values by using both the improved Liao et al.'s and Liao et al.'s weighted correlation coefficient approach in the same domain

The improved Chen et al.'s weighted correlation coefficient matrixes are

$$\rho_{\text{impw6}} = \begin{pmatrix} 1.0000 & 0.8215 & 0.6070 & 0.6550 & 0.7571 & 0.8686 & 0.6505 & 0.5990 & 0.5316 & 0.3171 \\ 0.8215 & 1.0000 & 0.6291 & 0.8243 & 0.8685 & 0.8844 & 0.8425 & 0.8364 & 0.6951 & 0.6209 \\ 0.6070 & 0.6291 & 1.0000 & 0.7229 & 0.6962 & 0.7487 & 0.1961 & 0.5476 & 0.8757 & 0.5257 \\ 0.6550 & 0.8243 & 0.7229 & 1.0000 & 0.8602 & 0.8420 & 0.5490 & 0.9023 & 0.8111 & 0.7829 \\ 0.7571 & 0.8685 & 0.6962 & 0.8602 & 1.0000 & 0.8767 & 0.6038 & 0.8323 & 0.6096 & 0.7761 \\ 0.8686 & 0.8844 & 0.7487 & 0.8420 & 0.8767 & 1.0000 & 0.6172 & 0.6895 & 0.6557 & 0.7021 \\ 0.6505 & 0.8425 & 0.1961 & 0.5490 & 0.6038 & 0.6172 & 1.0000 & 0.6029 & 0.3326 & 0.4334 \\ 0.5990 & 0.8364 & 0.5476 & 0.9023 & 0.8323 & 0.6895 & 0.6029 & 1.0000 & 0.7364 & 0.6225 \\ 0.5316 & 0.6951 & 0.8757 & 0.8111 & 0.6096 & 0.6557 & 0.3326 & 0.7364 & 1.0000 & 0.4604 \\ 0.3171 & 0.6209 & 0.5257 & 0.7829 & 0.7761 & 0.7021 & 0.4334 & 0.6225 & 0.4604 & 1.0000 \end{pmatrix}$$

$$\rho_{\text{impw7}} = \begin{pmatrix} 1.0000 & 0.4356 & 0.5946 & 0.4290 & 0.4153 & 0.7143 & 0.3852 & 0.5660 & 0.4560 & 0.1831 \\ 0.4356 & 1.0000 & 0.3406 & 0.6673 & 0.8396 & 0.5703 & 0.7545 & 0.4693 & 0.4297 & 0.5703 \\ 0.5946 & 0.3406 & 1.0000 & 0.4834 & 0.3898 & 0.6286 & 0.1185 & 0.5283 & 0.7668 & 0.3099 \\ 0.4290 & 0.6673 & 0.4834 & 1.0000 & 0.7203 & 0.6707 & 0.4963 & 0.6254 & 0.6193 & 0.6901 \\ 0.4153 & 0.8396 & 0.3898 & 0.7203 & 1.0000 & 0.5847 & 0.5593 & 0.4831 & 0.3898 & 0.7373 \\ 0.7143 & 0.5703 & 0.6286 & 0.6707 & 0.5847 & 1.0000 & 0.4444 & 0.6000 & 0.6286 & 0.4930 \\ 0.3852 & 0.7545 & 0.1185 & 0.4963 & 0.5593 & 0.4444 & 1.0000 & 0.3778 & 0.2296 & 0.4225 \\ 0.5660 & 0.4693 & 0.5283 & 0.6254 & 0.4831 & 0.6000 & 0.3778 & 1.0000 & 0.6684 & 0.3803 \\ 0.4560 & 0.4297 & 0.7668 & 0.6193 & 0.3898 & 0.6286 & 0.2296 & 0.6684 & 1.0000 & 0.3099 \\ 0.1831 & 0.5703 & 0.3099 & 0.6901 & 0.7373 & 0.4930 & 0.4225 & 0.3803 & 0.3099 & 1.0000 \end{pmatrix}$$

The improved Liao et al.'s weighted correlation coefficient matrixes are

$$\rho_{rw8} = \begin{pmatrix} 1.0000 & -0.7762 & -0.6050 & 0.0490 & -0.8078 & 0.9432 & 0.3322 & -0.7225 & -0.2176 & -0.3349 \\ -0.7762 & 1.0000 & 0.9658 & 0.5159 & 0.7006 & -0.7496 & 0.2940 & 0.6768 & 0.7655 & -0.3186 \\ -0.6050 & 0.9658 & 1.0000 & 0.6824 & 0.5778 & -0.5667 & 0.5302 & 0.6501 & 0.8877 & -0.5371 \\ 0.0490 & 0.5159 & 0.6824 & 1.0000 & -0.1969 & 0.1702 & 0.8517 & 0.0667 & 0.9344 & -0.9172 \\ -0.8078 & 0.7006 & 0.5778 & -0.1969 & 1.0000 & -0.9159 & -0.1828 & 0.7955 & 0.1570 & 0.2578 \\ 0.9432 & -0.7496 & -0.5667 & 0.1702 & -0.9159 & 1.0000 & 0.3679 & -0.6543 & -0.1558 & -0.3612 \\ 0.3322 & 0.2940 & 0.5302 & 0.8517 & -0.1828 & 0.3679 & 1.0000 & 0.1145 & 0.7922 & -0.9677 \\ -0.7225 & 0.6768 & 0.6501 & 0.0667 & 0.7955 & -0.6543 & 0.1145 & 1.0000 & 0.3072 & 0.0465 \\ -0.2176 & 0.7655 & 0.8877 & 0.9344 & 0.1570 & -0.1558 & 0.7922 & 0.3072 & 1.0000 & -0.8437 \\ -0.3349 & -0.3186 & -0.5371 & -0.9172 & 0.2578 & -0.3612 & -0.9677 & 0.0465 & -0.8437 & 1.0000 \end{pmatrix}$$

$$\rho_{rw9} = \begin{pmatrix} 1.0000 & -0.6698 & -0.3771 & 0.0258 & -0.7422 & 0.7870 & 0.2897 & -0.7151 & -0.1187 & -0.3097 \\ -0.6698 & 1.0000 & 0.6976 & 0.3147 & 0.5555 & -0.7249 & 0.2909 & 0.5901 & 0.4839 & -0.2543 \\ -0.3771 & 0.6976 & 1.0000 & 0.5763 & 0.3309 & -0.4233 & 0.3789 & 0.4094 & 0.7770 & -0.3096 \\ 0.0258 & 0.3147 & 0.5763 & 1.0000 & -0.0952 & 0.1074 & 0.5141 & 0.0354 & 0.9015 & -0.4465 \\ -0.7422 & 0.5555 & 0.3309 & -0.0952 & 1.0000 & -0.7022 & -0.1465 & 0.7234 & 0.0787 & 0.2561 \\ 0.7870 & -0.7249 & -0.4233 & 0.1074 & -0.7022 & 1.0000 & 0.3521 & -0.5516 & -0.1019 & -0.2788 \\ 0.2897 & 0.2909 & 0.3789 & 0.5141 & -0.1465 & 0.3521 & 1.0000 & 0.1009 & 0.4956 & -0.7805 \\ -0.7151 & 0.5901 & 0.4094 & 0.0354 & 0.7234 & -0.5516 & 0.1009 & 1.0000 & 0.1693 & 0.0426 \\ -0.1187 & 0.4839 & 0.7770 & 0.9015 & 0.0787 & -0.1019 & 0.4956 & 0.1693 & 1.0000 & -0.4257 \\ -0.3097 & -0.2543 & -0.3096 & -0.4465 & 0.2561 & -0.2788 & -0.7805 & 0.0426 & -0.4257 & 1.0000 \end{pmatrix}$$

Table 7 The evaluation hesitant fuzzy information for the 5 criteria of 10 firms

Firms	Managers' work experience	Profitability	Operating capacity	Debt-paying ability	Market competition
A_1	{0.3,0.4,0.5}	{0.4,0.5}	{0.8}	{0.5}	{0.2,0.3}
A_2	{0.4,0.6}	{0.6,0.8}	{0.2,0.3}	{0.3,0.4}	{0.6,0.7,0.9}
A_3	{0.5,0.7}	{0.9}	{0.3,0.4}	{0.3}	{0.8,0.9}
A_4	{0.3,0.4,0.5}	{0.8,0.9}	{0.7,0.9}	{0.1,0.2}	{0.9,1.0}
A_5	{0.8,1.0}	{0.8,1.0}	{0.4,0.6}	{0.8}	{0.7,0.8}
A_6	{0.4,0.5,0.6}	{0.2,0.3}	{0.9,1.0}	{0.5}	{0.3,0.4,0.5}
A_7	{0.6}	{0.7,0.9}	{0.8}	{0.3,0.4}	{0.4,0.7}
A_8	{0.9,1.0}	{0.7,0.8}	{0.4,0.5}	{0.5,0.6}	{0.7}
A_9	{0.4,0.6}	{1.0}	{0.6,0.7}	{0.2,0.3}	{0.9,1.0}
A_{10}	{0.9}	{0.6,0.7}	{0.5,0.8}	{1.0}	{0.7,0.8,0.9}

Step 2. Check whether these above four correlation coefficient matrixes are equivalent correlation coefficient matrixes by the examination rule. We notice that none of them satisfy $C^2 \subseteq C$, so we construct the equivalent correlation coefficient matrixes according to $C^{2^k}: C \to C^2 \to \cdots \to C^{2^k} \to \cdots$, until $C^{2^k} = C^{2^{k+1}}$, and obtain the equivalent correlation coefficient matrixes as follows:

The improved Chen et al.'s weighted equivalent correlation coefficient matrixes are

$$\rho^{8}_{\mathrm{impw6}} = \rho^{4}_{\mathrm{impw6}} \circ \rho^{4}_{\mathrm{impw6}} = \begin{pmatrix} 1.0000 & 0.8686 & 0.8111 & 0.8602 & 0.8686 & 0.8686 & 0.8425 & 0.8602 & 0.8111 & 0.7829 \\ 0.8686 & 1.0000 & 0.8111 & 0.8602 & 0.8767 & 0.8844 & 0.8425 & 0.8602 & 0.8111 & 0.7829 \\ 0.8111 & 0.8111 & 1.0000 & 0.8111 & 0.8111 & 0.8111 & 0.8111 & 0.8111 & 0.8757 & 0.7829 \\ 0.8602 & 0.8602 & 0.8111 & 1.0000 & 0.8602 & 0.8602 & 0.8425 & 0.9023 & 0.8111 & 0.7829 \\ 0.8686 & 0.8767 & 0.8111 & 0.8602 & 1.0000 & 0.8767 & 0.8425 & 0.8602 & 0.8111 & 0.7829 \\ 0.8686 & 0.8844 & 0.8111 & 0.8602 & 0.8767 & 1.0000 & 0.8425 & 0.8602 & 0.8111 & 0.7829 \\ 0.8425 & 0.8425 & 0.8111 & 0.8425 & 0.8425 & 0.8425 & 1.0000 & 0.8425 & 0.8111 & 0.7829 \\ 0.8602 & 0.8602 & 0.8111 & 0.9023 & 0.8602 & 0.8602 & 0.8425 & 1.0000 & 0.8111 & 0.7829 \\ 0.8111 & 0.8111 & 0.8757 & 0.8111 & 0.8111 & 0.8111 & 0.8111 & 0.8111 & 1.0000 & 0.7829 \\ 0.7829 & 0.7829 & 0.7829 & 0.7829 & 0.7829 & 0.7829 & 0.7829 & 0.7829 & 0.7829 & 1.0000 \end{pmatrix}$$

$$\rho^{16}_{\mathrm{impw7}} = \rho^{8}_{\mathrm{impw7}} \circ \rho^{8}_{\mathrm{impw7}} = \begin{pmatrix} 1.0000 & 0.6707 & 0.6286 & 0.6707 & 0.6707 & 0.7143 & 0.6707 & 0.6286 & 0.6286 & 0.6707 \\ 0.6707 & 1.0000 & 0.6286 & 0.7203 & 0.8396 & 0.6707 & 0.7545 & 0.6286 & 0.6286 & 0.7373 \\ 0.6286 & 0.6286 & 1.0000 & 0.6286 & 0.6286 & 0.6286 & 0.6286 & 0.6684 & 0.7668 & 0.6286 \\ 0.6707 & 0.7203 & 0.6286 & 1.0000 & 0.7203 & 0.6707 & 0.7203 & 0.6286 & 0.6286 & 0.7203 \\ 0.6707 & 0.8396 & 0.6286 & 0.7203 & 1.0000 & 0.6707 & 0.7545 & 0.6286 & 0.6286 & 0.7373 \\ 0.7143 & 0.6707 & 0.6286 & 0.6707 & 0.6707 & 1.0000 & 0.6707 & 0.6286 & 0.6286 & 0.6707 \\ 0.6707 & 0.7545 & 0.6286 & 0.7203 & 0.7545 & 0.6707 & 1.0000 & 0.6286 & 0.6286 & 0.7373 \\ 0.6286 & 0.6286 & 0.6684 & 0.6286 & 0.6286 & 0.6286 & 0.6286 & 1.0000 & 0.6684 & 0.6286 \\ 0.6286 & 0.6286 & 0.7668 & 0.6286 & 0.6286 & 0.6286 & 0.6286 & 0.6684 & 1.0000 & 0.6286 \\ 0.6707 & 0.7373 & 0.6286 & 0.7203 & 0.7373 & 0.6707 & 0.7373 & 0.6286 & 0.6286 & 1.0000 \end{pmatrix}$$

Table 8 The clustering result of 10 firms with respect to the improved Chen et al.'s correlation coefficient (44)

Class	Confidence level	Clusters
10	$0.9023 < \lambda \leq 1$	$\{A_1\}$, $\{A_2\}$, $\{A_3\}$, $\{A_4\}$, $\{A_5\}$, $\{A_6\}$, $\{A_7\}$, $\{A_8\}$, $\{A_9\}$, $\{A_{10}\}$
9	$0.8844 < \lambda \leq 0.9023$	$\{A_1\}$, $\{A_2\}$, $\{A_3\}$, $\{A_4, A_8\}$, $\{A_5\}$, $\{A_6\}$, $\{A_7\}$, $\{A_9\}$, $\{A_{10}\}$
8	$0.8767 < \lambda \leq 0.8844$	$\{A_1\}$, $\{A_2, A_6\}$, $\{A_3\}$, $\{A_4, A_8\}$, $\{A_5\}$, $\{A_7\}$, $\{A_9\}$, $\{A_{10}\}$
7	$0.8757 < \lambda \leq 0.8767$	$\{A_1\}$, $\{A_2, A_5, A_6\}$, $\{A_3\}$, $\{A_4, A_8\}$, $\{A_7\}$, $\{A_9\}$, $\{A_{10}\}$
6	$0.8686 < \lambda \leq 0.8757$	$\{A_1\}$, $\{A_2, A_5, A_6\}$, $\{A_3, A_9\}$, $\{A_4, A_8\}$, $\{A_7\}$, $\{A_{10}\}$
5	$0.8602 < \lambda \leq 0.8686$	$\{A_1, A_4, A_8\}$, $\{A_2, A_5, A_6\}$, $\{A_3, A_9\}$, $\{A_7\}$, $\{A_{10}\}$
4	$0.8425 < \lambda \leq 0.8602$	$\{A_1, A_2, A_4, A_5, A_6, A_8\}$, $\{A_3, A_9\}$, $\{A_7\}$, $\{A_{10}\}$
3	$0.8111 < \lambda \leq 0.8425$	$\{A_1, A_2, A_4, A_5, A_6, A_7, A_8\}$, $\{A_3, A_9\}$, $\{A_{10}\}$
2	$0.7829 < \lambda \leq 0.8111$	$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9\}$, $\{A_{10}\}$
1	$0 < \lambda \leq 0.7829$	$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}\}$

Table 9 The clustering result of 10 firms with respect to the improved Chen et al.'s correlation coefficient (45)

Class	Confidence level	Clusters
10	$0.8396<\lambda\leq 1$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_9\}, \{A_{10}\}$
9	$0.7668<\lambda\leq 0.8396$	$\{A_1\}, \{A_2, A_5\}, \{A_3\}, \{A_4\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_9\}, \{A_{10}\}$
8	$0.7545<\lambda\leq 0.7668$	$\{A_1\}, \{A_2, A_5\}, \{A_3, A_9\}, \{A_4\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_{10}\}$
7	$0.7373<\lambda\leq 0.7545$	$\{A_1\}, \{A_2, A_5, A_7\}, \{A_3, A_9\}, \{A_4\}, \{A_6\}, \{A_8\}, \{A_{10}\}$
6	$0.7203<\lambda\leq 0.7373$	$\{A_1\}, \{A_2, A_5, A_7, A_{10}\}, \{A_3, A_9\}, \{A_4\}, \{A_6\}, \{A_8\}$
5	$0.7143<\lambda\leq 0.7203$	$\{A_1\}, \{A_2, A_4, A_5, A_7, A_{10}\}, \{A_3, A_9\}, \{A_6\}, \{A_8\}$
4	$0.6707<\lambda\leq 0.7143$	$\{A_1, A_2, A_4, A_5, A_7, A_{10}\}, \{A_3, A_9\}, \{A_6\}, \{A_8\}$
3	$0.6684<\lambda\leq 0.6707$	$\{A_1, A_2, A_4, A_5, A_6, A_7, A_{10}\}, \{A_3, A_9\}, \{A_8\}$
2	$0.6286<\lambda\leq 0.6684$	$\{A_1, A_2, A_4, A_5, A_6, A_7, A_{10}\}, \{A_3, A_8, A_9\}$
1	$0<\lambda\leq 0.6286$	$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}\}$

The improved Liao et al.'s weighted equivalent correlation coefficient matrixes are

$$\rho^8_{rw8}=\rho^4_{rw8}\circ\rho^4_{rw8}=\begin{pmatrix}
1.0000 & 0.3679 & 0.3679 & 0.3679 & 0.3679 & 0.9432 & 0.3679 & 0.3679 & 0.3679 & 0.2578\\
0.3679 & 1.0000 & 0.9658 & 0.8877 & 0.7006 & 0.3679 & 0.8517 & 0.7006 & 0.8877 & 0.2578\\
0.3679 & 0.9658 & 1.0000 & 0.8877 & 0.7006 & 0.3679 & 0.8517 & 0.7006 & 0.8877 & 0.2578\\
0.3679 & 0.8877 & 0.8877 & 1.0000 & 0.7006 & 0.3679 & 0.8517 & 0.7006 & 0.9344 & 0.2578\\
0.3679 & 0.7006 & 0.7006 & 0.7006 & 1.0000 & 0.3679 & 0.7006 & 0.7955 & 0.7006 & 0.2578\\
0.9432 & 0.3679 & 0.3679 & 0.3679 & 0.3679 & 1.0000 & 0.3679 & 0.3679 & 0.3679 & 0.2578\\
0.3679 & 0.8517 & 0.8517 & 0.8517 & 0.7006 & 0.3679 & 1.0000 & 0.7006 & 0.8517 & 0.2578\\
0.3679 & 0.7006 & 0.7006 & 0.7006 & 0.7955 & 0.3679 & 0.7006 & 1.0000 & 0.7006 & 0.2578\\
0.3679 & 0.8877 & 0.8877 & 0.9344 & 0.7006 & 0.3679 & 0.8517 & 0.7006 & 1.0000 & 0.2578\\
0.2578 & 0.2578 & 0.2578 & 0.2578 & 0.2578 & 0.2578 & 0.2578 & 0.2578 & 0.2578 & 1.0000
\end{pmatrix}$$

$$\rho^{16}_{rw9}=\rho^8_{rw9}\circ\rho^8_{rw9}=\begin{pmatrix}
1.0000 & 0.3521 & 0.3521 & 0.3521 & 0.3521 & 0.7870 & 0.3521 & 0.3521 & 0.3521 & 0.2561\\
0.3521 & 1.0000 & 0.6976 & 0.6976 & 0.5901 & 0.3521 & 0.5141 & 0.5901 & 0.6976 & 0.2561\\
0.3521 & 0.6976 & 1.0000 & 0.7770 & 0.5901 & 0.3521 & 0.5141 & 0.5901 & 0.7770 & 0.2561\\
0.3521 & 0.6976 & 0.7770 & 1.0000 & 0.5901 & 0.3521 & 0.5141 & 0.5901 & 0.9015 & 0.2561\\
0.3521 & 0.5901 & 0.5901 & 0.5901 & 1.0000 & 0.3521 & 0.5141 & 0.7234 & 0.5901 & 0.2561\\
0.7870 & 0.3521 & 0.3521 & 0.3521 & 0.3521 & 1.0000 & 0.3521 & 0.3521 & 0.3521 & 0.2561\\
0.3521 & 0.5141 & 0.5141 & 0.5141 & 0.5141 & 0.3521 & 1.0000 & 0.5141 & 0.5141 & 0.2561\\
0.3521 & 0.5901 & 0.5901 & 0.5901 & 0.7234 & 0.3521 & 0.5141 & 1.0000 & 0.5901 & 0.2561\\
0.3521 & 0.6976 & 0.7770 & 0.9015 & 0.5901 & 0.3521 & 0.5141 & 0.5901 & 1.0000 & 0.2561\\
0.2561 & 0.2561 & 0.2561 & 0.2561 & 0.2561 & 0.2561 & 0.2561 & 0.2561 & 0.2561 & 1.0000
\end{pmatrix}$$

Table 10 The clustering result of 10 firms with respect to the improved Liao et al.'s correlation coefficient (48)

Class	Confidence level	Clusters
10	$0.9015<\lambda\leq 1$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_9\}, \{A_{10}\}$
9	$0.7870<\lambda\leq 0.9015$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4, A_9\}, \{A_5\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_{10}\}$
8	$0.7770<\lambda\leq 0.7870$	$\{A_1, A_6\}, \{A_2\}, \{A_3\}, \{A_4, A_9\}, \{A_5\}, \{A_7\}, \{A_8\}, \{A_{10}\}$
7	$0.7234<\lambda\leq 0.7770$	$\{A_1, A_6\}, \{A_2\}, \{A_3, A_4, A_9\}, \{A_5\}, \{A_7\}, \{A_8\}, \{A_{10}\}$
6	$0.6976<\lambda\leq 0.7234$	$\{A_1, A_6\}, \{A_2\}, \{A_3, A_4, A_9\}, \{A_5, A_8\}, \{A_7\}, \{A_{10}\}$
5	$0.5901<\lambda\leq 0.6976$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_9\}, \{A_5, A_8\}, \{A_7\}, \{A_{10}\}$
4	$0.5141<\lambda\leq 0.5901$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_5, A_8, A_9\}, \{A_7\}, \{A_{10}\}$
3	$0.3521<\lambda\leq 0.5141$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_5, A_7, A_8, A_9\}, \{A_{10}\}$
2	$0.2561<\lambda\leq 0.3521$	$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9\}, \{A_{10}\}$
1	$0<\lambda\leq 0.2561$	$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}\}$

Table 11 The clustering result of 10 firms with respect to the improved Liao et al.'s correlation coefficient (49)

Class	Confidence level	Clusters
10	$0.9810 < \lambda \le 1$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_9\}, \{A_{10}\}$
9	$0.9761 < \lambda \le 0.9810$	$\{A_1\}, \{A_2, A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_9\}, \{A_{10}\}$
8	$0.9585 < \lambda \le 0.9761$	$\{A_1, A_6\}, \{A_2, A_3\}, \{A_4\}, \{A_5\}, \{A_7\}, \{A_8\}, \{A_9\}, \{A_{10}\}$
7	$0.9490 < \lambda \le 0.9585$	$\{A_1, A_6\}, \{A_2, A_3\}, \{A_4, A_9\}, \{A_5\}, \{A_7\}, \{A_8\}, \{A_{10}\}$
6	$0.9475 < \lambda \le 0.9490$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_9\}, \{A_5\}, \{A_7\}, \{A_8\}, \{A_{10}\}$
5	$0.9189 < \lambda \le 0.9475$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_9\}, \{A_5, A_8\}, \{A_7\}, \{A_{10}\}$
4	$0.8974 < \lambda \le 0.9189$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_7, A_9\}, \{A_5, A_8\}, \{A_{10}\}$
3	$0.8004 < \lambda \le 0.8974$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_5, A_7, A_8, A_9\}, \{A_{10}\}$
2	$0.7091 < \lambda \le 0.8004$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_5, A_7, A_8, A_9, A_{10}\}$
1	$0 < \lambda \le 0.7091$	$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}\}$

Table 12 The clustering result of 10 firms with respect to Chen et al.'s correlation coefficient

Class	Confidence level	Clusters
10	$0.9515 < \lambda \le 1$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_9\}, \{A_{10}\}$
9	$0.9306 < \lambda \le 0.9515$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5, A_{10}\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_9\}$
8	$0.9238 < \lambda \le 0.9306$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4, A_9\}, \{A_5, A_{10}\}, \{A_6\}, \{A_7\}, \{A_8\}$
7	$0.9104 < \lambda \le 0.9238$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4, A_7, A_9\}, \{A_5, A_{10}\}, \{A_6\}, \{A_8\}$
6	$0.9025 < \lambda \le 0.9104$	$\{A_1, A_6\}, \{A_2\}, \{A_3\}, \{A_4, A_7, A_9\}, \{A_5, A_{10}\}, \{A_8\}$
5	$0.8997 < \lambda \le 0.9025$	$\{A_1, A_6\}, \{A_2\}, \{A_3\}, \{A_4, A_7, A_8, A_9\}, \{A_5, A_{10}\}$
4	$0.8520 < \lambda \le 0.8997$	$\{A_1, A_6\}, \{A_2\}, \{A_3, A_4, A_7, A_8, A_9\}, \{A_5, A_{10}\}$
3	$0.8200 < \lambda \le 0.8520$	$\{A_1, A_6\}, \{A_2\}, \{A_3, A_4, A_5, A_7, A_8, A_9, A_{10}\}$
2	$0.7984 < \lambda \le 0.8200$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_5, A_7, A_8, A_9, A_{10}\}$
1	$0 < \lambda \le 0.7984$	$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}\}$

That is to say ρ^4_{impw6}, ρ^8_{impw7}, ρ^4_{rw8}, and ρ^8_{rw9} are the equivalent correlation coefficient matrixes.

Step 3. For a given confidence level $\lambda \in [0, 1]$, we can construct the λ-cutting matrix $C_\lambda = ({}_\lambda\rho_{ij})_{m \times m}$ to classify these HFSs by the classify principle. The possible classifications of these firms by these improved correlation coefficients are shown in Tables 8, 9, 10, and 11.

We also classify them with Chen et al.'s and Liao et al.'s correlation coefficients to compare with the improved correlation coefficients, which are shown in Table 12 and Table 13.

Comparing the cluster results from Table 8, 9, 10, 11, 12 and 13, we can get different results. The confidence level and clustering process are different from each other. We cannot determine which clustering result is the best, because they are all correct to a certain extent. They are from different

Table 13 The clustering result of 10 firms with respect to Liao et al.'s correlation coefficient

Class	Confidence level	Clusters
10	$0.9659 < \lambda \le 1$	$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_9\}, \{A_{10}\}$
9	$0.9365 < \lambda \le 0.9659$	$\{A_1\}, \{A_2, A_3\}, \{A_4\}, \{A_5\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_9\}, \{A_{10}\}$
8	$0.8665 < \lambda \le 0.9365$	$\{A_1\}, \{A_2, A_3\}, \{A_4, A_9\}, \{A_5\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_{10}\}$
7	$0.8272 < \lambda \le 0.8665$	$\{A_1\}, \{A_2, A_3, A_4, A_9\}, \{A_5\}, \{A_6\}, \{A_7\}, \{A_8\}, \{A_{10}\}$
6	$0.8068 < \lambda \le 0.8272$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_9\}, \{A_5\}, \{A_7\}, \{A_8\}, \{A_{10}\}$
5	$0.7073 < \lambda \le 0.8068$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_9\}, \{A_5, A_8\}, \{A_7\}, \{A_{10}\}$
4	$0.6062 < \lambda \le 0.7073$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_7, A_9\}, \{A_5, A_8\}, \{A_{10}\}$
3	$0.4225 < \lambda \le 0.6062$	$\{A_1, A_6\}, \{A_2, A_3, A_4, A_5, A_7, A_8, A_9\}, \{A_{10}\}$
2	$0.3949 < \lambda \le 0.4225$	$\{A_1, \{A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9\}, \{A_{10}\}$
1	$0 < \lambda \le 0.3949$	$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}\}$

definition of views. However, the improved correlation coefficients modify the drawbacks in Chen et al.'s and Liao et al.'s correlation coefficients, which deserve to be more convincing and accurate.

Conclusions

In this paper, we focus on the correlation coefficients of HFSs. We show that some definitions of the existing correlation coefficients are counter-intuitive, so we provide the improved versions in the view of the mathematics and stochastic process rules to make them more intuitive. Besides, we use two examples about medical diagnosis and cluster analysis to compare the improved weighted correlation coefficient with the existing correlation coefficients. The comparison results illustrate that the improved weighted correlation coefficients are better than Chen et al's and Liao et al.'s weighted correlation coefficient formula both in accuracy and in discrimination.

In the future, the idea of this paper in improving the correlation coefficients are expected to be used in the improvement of the correlations and correlation coefficients for other forms of fuzzy sets, such as the intuitionistic fuzzy, dual hesitant fuzzy sets, hesitant probabilistic fuzzy sets, and neutrosophic hesitant fuzzy sets. Moreover, novel correlation coefficients with this notion which can additionally relax the length and order of the memberships in the HFSs remain to be explored.

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Compliance with Ethical Standards

Conflict of Interest All the authors declare that they have no conflict of interest.

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors.

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