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Spatial Temporal Dynamic of a Coupled Reaction-Diffusion Neural Network with Time Delay

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Abstract
In neural networks, the diffusion effect cannot be avoided due to the electrons diffuse from the high region to low region. However, the spatial temporal dynamic of neural network with diffusion and time delay is not well understood. The goal of this paper is to study the spatial temporal dynamic of a coupled neural network with diffusion and time delay. Based on the eigenvalue of the Laplace operator, the characteristic equation is obtained. By analyzing the characteristic equation, some conditions for the occurrence of Turing instability and Hopf bifurcations are obtained. Moreover, normal form theory and center manifold theorem of the partial differential equation are used to analyze the period and direction of Hopf bifurcation. It found that the diffusion coefficients can lead to the diffusion-driven instability, and time delay can give rise to the periodic solution. Near the Turing instability point, there exist some spatially non-homogeneous patterns such as spike, spiral wave, and zebra-stripe. Near the Hopf bifurcation point, the spatial temporal dynamic can be divided into four types: the stable zero equilibrium, the two distinct stripe patterns, and the irregular pattern. The effects of diffusion and time delay on the spatial temporal dynamic of a coupled reaction-diffusion neural network with time delay are investigated. It is found that the diffusion coefficients have a marked impact on selection of the type and characteristics of the emerging pattern. The results obtained in this paper are novel and supplement some existing works.

Keywords Reaction-diffusion neuron network · Turing instability · Hopf bifurcation · Pattern formation

Introduction

Recently, the reaction-diffusion neural networks (RDNNs) have attracted much attention, for it has been applied in many fields such as diffusion circuit design, signal processing, as well as system optimization [1–8]. As is known to all, time delay is prevalent in RDNNs, for it takes time to transmit signal in synapse. Time delay makes the dynamic behaviors of RDNNs more complicated, e.g., altering the patterns of equilibria, causing bifurcations, and chaos. Consequently, studies of dynamic behaviors of RDNNs with time delay have become intensive. A number of significant results can be found from the literatures [9–18].

In practical reaction-diffusion systems, the Turing instability is one of famous spatial temporal dynamic phenomena, which is found by Turing in 1952. The Turing instability means that the steady state of system is stable without diffusion and becomes unstable when the diffusion terms are added into the system. The Turing instability can lead to the diffusion-driven instability and the existence of spatial non-homogeneous patterns such as spike, spiral wave, and zebra-stripe. To date, the spatial temporal dynamics based on the Turing instability have been investigated in many systems such as chemical systems, ecological systems, and predator-prey systems [19–23]. However, to best of the authors' knowledge, few works investigate the spatial temporal dynamic of neural networks such as Turing instability and Hopf bifurcation. The observation gives us the motivation to investigate the spatial temporal dynamic of neural network with time delay.

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Inspired by the above discussions, in this paper, we investigate the spatial temporal dynamic for a class of delayed RDNN with homogeneous Neumann boundary conditions. The model of the RDNN can be described as follows:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = d_1 \Delta u(t,x) - c_1 u(t,x) + b_1 f_1[u(t,x)] + a_1 f_2[v(t-\tau,x)], t>0, x\in\Omega, \\ \frac{\partial v(t,x)}{\partial t} = d_2 \Delta v(t,x) - c_2 v(t,x) + b_2 f_3[v(t,x)] + a_2 f_4[u(t-\tau,x)], t>0, x\in\Omega, \end{cases} \quad (1.1)$$

with

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0, \qquad t>0, x\in\partial\Omega,$$
$$u(t,x) = \phi_1(t,x), \quad t\in[-\tau,0], x\in\overline{\Omega},$$
$$v(t,x) = \phi_2(t,x), \quad t\in[-\tau,0], x\in\overline{\Omega},$$

where u, v is the neuron state at time $t>0$, $\Omega=(0,\pi)$ is the bounded domain, and Δ is the Laplacian operator; d_1, $d_2>0$ are the diffusion coefficients, τ is the time delay, $c_i>0 (i=1,2)$ denote the self-feedback rate of the neuron, $a_i (i=1,2)$ is the connection weight, $b_i (i=1,2)$ are self-connection weight, and $f_i (i=1,2,3,4)$ is the activation functions of neurons with $f_i(0)=0$ and $f_i'(0)=1$. There are three contributions of this paper, which are as follows:

1. The conditions of Turing instability with and without time delay are obtained. Our conditions show that the diffusion coefficients can lead to the diffusion-driven instability and the occurrence of stripe spatial pattern.
2. Comparing with the existing works, some novel conditions where the Hopf bifurcation occurs based on the Turing instability are obtained. The spatial temporal dynamic near the Hopf bifurcation point can be divided into four types: the stable zero equilibrium, the two distinct stripe patterns, and the irregular pattern.
3. The numerical results not only validate the obtained theorems but also show that the diffusion coefficients have a marked impact on selection of the type and characteristics of the emerging pattern. With the diffusion coefficients increasing, different patterns appear.

The rest of the paper is organized as follows. In the “Turing Instability Analysis” section, the conditions of Turing instability without time delay are obtained. In the “Bifurcation Analysis” section, the conditions of Hopf bifurcation are obtained. Moreover, we give the conditions of Turing instability with time delay. In “The Normal Form of Hopf Bifurcation” section, the normal form and the direction of Hopf bifurcation are obtained. In the “Numerical Examples” section, numerical results not only validate the obtained theorems but also show that the diffusion coefficients have a marked impact on pattern formation.

Turing Instability Analysis

In this section, the conditions of Turing instability of (1.1) without time delay are presented. As $f_i(0)=0$, one can obtain that $(0,0)^T$ is the equilibrium of system (1.1). Let $U(t)=(u(t,\cdot), v(t,\cdot))^T=(u(t), v(t))^T$, the linear part of the system (1.1) at $(0,0)^T$ can be expressed as:

$$\dot{U}(t) = D\Delta U(t) + L(U_t), \quad (2.1)$$

where $D=\text{diag}\{d_1,d_2\}$, $L: C\to R^2$. The characteristic equation of (2.1) is

$$\lambda y - D\Delta y - L(e^{\lambda} y) = 0, \quad y\in dom(\Delta). \quad (2.2)$$

Define $-k^2 (k\in K_0, K_0=\{1,2,3,...\})$ as the eigenvalue of Δ on X, following the method of [17], (2.2) can be rewritten as:

$$F(\lambda) = \lambda^2 + A_k\lambda + B_k + Ce^{-2\lambda\tau} = 0, k\in K_0, \quad (2.3)$$

where

$$b_{11} = b_1 f_1'(0),\ a_{11} = a_1 f_2'(0),\ b_{22} = b_2 f_3'(0),\ a_{22} = a_2 f_4'(0),$$
$$A_k = (d_1+d_2)k^2 + c_1 + c_2 - b_{11} - b_{22},$$
$$B_k = d_1 d_2 k^4 + ((c_2-b_{22})d_1 + (c_1-b_{11})d_2)k^2 + (c_1-b_{11})(c_2-b_{22}),$$
$$C = -a_{11}a_{22}.$$

For $\tau=0$, (2.3) becomes

$$F(\lambda) = \lambda^2 + A_k\lambda + B_k + C, k\in K_0, \quad (2.4)$$

and the roots of (2.4) is given by

$$\lambda_{1,2} = \frac{-A_k \pm \sqrt{(A_k)^2 - 4(B_k+C)}}{2}, k\in K_0. \quad (2.5)$$

Let $c_0=b_{11}+b_{22}-c_1$, it is easy to see that if $c_2>c_0$ and $(c_1-b_{11})^2+a_{11}a_{22}<0$, we can obtain $A_0>0$ and $B_0+C>0$, which means $\text{Re}\{\lambda_{1,2}\}<0$ with $k=0$. Thus, the original equilibrium of system (1.1) without diffusion and time delay is stable if $(c_1-b_{11})^2+a_{11}a_{22}<0$ and $c_2>c_0$ are satisfied. From [19], we know Turing instability may occur if the roots of (2.4) have positive parts under $(c_1-b_{11})^2+a_{11}a_{22}<0$ and $c_2>c_0$, that is, there exists $k\in K_0$ such that $A_k<0$ or $B_k+C<$

0. As $A_k > c_1 + c_2 - b_{11} - b_{22}$ for any $k \in K_0$, one can obtain that $A_k > 0$ if $c_2 > c_0$. Note that $B_0 + C > 0$ when $c_2 > c_0$ and $(c_1 - b_{11})^2 + a_{11}a_{22} < 0$ are satisfied, we can deduce that Turing instability occurs if there exists $k \in K_0$ such that $B_k + C < 0$.

Now, we consider the following function:

$$\begin{aligned} G(y) &= d_1 d_2 y^2 + ((c_2 - b_{22})d_1 + (c_1 - b_{11})d_2)y \\ &\quad + (c_1 - b_{11})(c_2 - b_{22}) - a_{11}a_{22} \\ &= 0, y > 0. \end{aligned} \tag{2.6}$$

Differentiating (2.6) with respect to y, one can obtain

$$G'(y) = 2d_1 d_2 y + (c_2 - b_{22})d_1 + (c_1 - b_{11})d_2 = 0. \tag{2.7}$$

By $G'(y) = 0$, we have

$$\overline{y} = -\frac{1}{2}\frac{(c_2 - b_{22})d_1 + (c_1 - b_{11})d_2}{d_1 d_2}, \tag{2.8}$$

from (2.6) to (2.8), we have if $\overline{y} > 0$ and $G(\overline{y}) < 0$ hold, there exists $y > 0$ such that $G(y) < 0$, which means $B_k + C < 0$. By simple calculation, the sufficient conditions for the occurrence of Turing instability are given by:

$$\begin{cases} \dfrac{(c_2 - b_{22})d_1 + (c_1 - b_{11})d_2}{d_1 d_2} < 0, \\ \Theta = ((c_2 - b_{22})d_1 + (c_1 - b_{11})d_2)^2 - 4d_1 d_2 (B_0 + C_0) > 0. \end{cases} \tag{2.9}$$

If $c_1 > b_{11}$, $c_2 > b_{22}$, the inequality of (2.9) is not satisfied, which means the Turing instability in this case does not exist. Thus, we only consider the case $c_2 > b_{22}$, $c_1 < b_{11}$ and $c_2 < b_{22}$, $c_1 > b_{11}$; by (2.9), we have

$$\begin{cases} \dfrac{d_1}{d_2} < d_{11} = -\dfrac{c_1 - b_{11}}{c_2 - b_{22}} \quad (c_2 > b_{22}, c_1 < b_{11}), \\ \dfrac{d_1}{d_2} > d_{22} = -\dfrac{c_1 - b_{11}}{c_2 - b_{22}} \quad (c_2 < b_{22}, c_1 > b_{11}). \end{cases} \tag{2.10}$$

In the following, we investigate the conditions where the second inequality of (2.9) is satisfied. Let

$$F(d_1) = ad_1^2 + bd_1 + c = 0, \tag{2.11}$$

where

$$\begin{aligned} a &= (c_2 - b_{22})^2 > 0, \\ b &= 2d_2(2a_{11}a_{22} - (c_1 - b_{11})(c_2 - b_{22})), \\ c &= (c_1 - b_{11})^2 d_2^2 > 0. \end{aligned}$$

It is easy to see that (2.11) has two positive real roots when $(b_{11} - c_1)^2 + a_{11}a_{22} < 0$ and $c_2 > c_0$ are satisfied; by calculation, we have

$$\begin{aligned} \left(\frac{d_1}{d_2}\right)^{\pm} &= -\frac{(2a_{11}a_{22} - (b_{22} - c_2)(b_{11} - c_1))}{(c_2 - b_2)^2} \\ &\quad \pm \frac{2\sqrt{a_{11}{}^2 a_{22}{}^2 - a_{11}a_{22}(b_{22} - c_2)(b_{11} - c_1)}}{(c_2 - b_2)^2} \end{aligned} \tag{2.12}$$

Let

$$\gamma_{1,2} = \frac{-(2a_{11}a_{22} - (b_{22} - c_2)(b_{11} - c_1))}{(c_2 - b_{22})^2} \mp \frac{2\sqrt{a_{11}{}^2 a_{22}{}^2 - a_{11}a_{22}(b_{22} - c_2)(b_{11} - c_1)}}{(c_2 - b_{22})^2},$$

since a, $c > 0$; by (2.12), we can obtain the parameter space of $\Theta > 0$ as follows

$$\frac{d_1}{d_2} \in (0, \gamma_1) \cup (\gamma_2, +\infty). \tag{2.13}$$

Combine with (2.10), we have

$$\begin{aligned} \gamma_1 - d_{11} &= -2\left(\frac{a_{11}a_{22} - (b_{22} - c_2)(b_{11} - c_1) + \sqrt{a_{11}{}^2 a_{22}{}^2 - a_{11}a_{22}(b_{22} - c_2)(b_{11} - c_1)}}{(c_2 - b_{22})^2}\right) < 0, \\ \gamma_2 - d_{11} &= -2\left(\frac{a_{11}a_{22} - (b_{22} - c_2)(b_{11} - c_1) - \sqrt{a_{11}{}^2 a_{22}{}^2 - a_{11}a_{22}(b_{22} - c_2)(b_{11} - c_1)}}{(c_2 - b_{22})^2}\right) > 0. \end{aligned} \tag{2.14}$$

since $a_{11}a_{22} - (b_{22} - c_2)(b_{11} - c_1) < 0$; from (2.14), we can obtain

$$\gamma_1 < d_{11}(d_{22}) < \gamma_2 \tag{2.15}$$

From the above discussion, we have the following conclusion:

Theorem 1 *For $\tau = 0$, suppose $(b_{11} - c_1)^2 + a_{11}a_{22} < 0$ and $c_2 > c_0$hold.*

(1) *If $c_2 > b_{22}$, $c_1 > b_{11}$, the system (1.1) is locally stable at zero equilibrium*;
(2) *If $c_2 > b_{22}$, $c_1 < b_{11}$* and *$\frac{d_1}{d_2} \in (\gamma_1, +\infty)$, the system (1.1) is locally stable at zero equilibrium*;
(3) *If $c_2 > b_{22}$, $c_1 < b_{11}$and $\frac{d_1}{d_2} \in (0, \gamma_1)$, the system (1.1) without diffusion is local stable at zero equilibrium. It becomes unstable with diffusion at zero equilibrium. That is, the zero equilibrium is Turing unstable.*
(4) *If $c_2 < b_{22}$, $c_1 > b_{11}$and $\frac{d_1}{d_2} \in (0, \gamma_2)$, the system (1.1) is locally stable at zero equilibrium*;
(5) *If $c_2 < b_{22}$, $c_1 > b_{11}$and $\frac{d_1}{d_2} \in (\gamma_2, +\infty)$, the system (1.1) without diffusion is local stable at zero equilibrium. It becomes unstable with diffusion at zero equilibrium. That is, the zero equilibrium is Turing unstable.*

Remark 1 In [18], the stability conditions of system (1.1) with $\tau = 0$ is $c_2 > b_{22}$, $c_1 > b_{11}$. From theorem 1, we can see the stability condition of [18] is a subset of our results.

Bifurcation Analysis

From Theorem 1, we can see that if one of the following conditions is satisfied, the system (1.1) is local stable with $\tau = 0$.

(H1) $(b_{11}-c_1)^2 + a_{11}a_{22} < 0, c_2 > b_{22}, c_1 < b_{11}, c_2 > c_0, \frac{d_1}{d_2} \in (\gamma_1, +\infty)$

(H2) $(b_{11}-c_1)^2 + a_{11}a_{22} < 0, c_2 < b_{22}, c_1 > b_{11}, c_2 > c_0, \frac{d_1}{d_2} \in (0, \gamma_2)$

(H3) $c_2 > b_{22}$, $c_1 > b_{11}$.

Now, we check the existence of Hopf bifurcation, which means (2.3) has $\pm i\omega$ roots. Suppose $\pm i\omega(\omega > 0)$ is a root of (2.3), we have

$$-\omega^2 + A_k i\omega + B_k + Ce^{-2i\omega\tau} = 0 \tag{3.1}$$

Separating the real and imaginary parts of (3.1), one can obtain

$$\begin{cases} -\omega^2 + B_k + C\cos 2\omega\tau = 0, \\ A_k\omega - C\sin 2\omega\tau = 0, \end{cases} \tag{3.2}$$

from (3.2), we can obtain

$$\begin{cases} \cos 2\omega\tau = \dfrac{\omega^2 - B_k}{C}, \\ \sin 2\omega\tau = \dfrac{A_k\omega}{C}, \end{cases} \tag{3.3}$$

from (3.3), we have

$$\omega^4 + \left({A_k}^2 - 2B_k\right)\omega^2 + B_k^2 - C^2 = 0, \tag{3.4}$$

where

$$\begin{aligned} A_k^2 - 2B_k &= \left(d_1^2 + d_2^2\right)k^4 + 2(d_1(c_1-b_{11}) + d_2(c_2-b_{22}))k^2 + (c_1-b_{11})^2 + (c_2-b_{22})^2, \\ B_k^2 - C^2 &= d_1^2 d_2^2 k^8 + 2d_1d_2((c_2-b_{22})d_1 + (c_1-b_{11})d_2)k^6 \\ &\quad + \left(((c_2-b_{22})d_1 + (c_1-b_{11})d_2)^2 + 2d_1d_2(c_1-b_{11})(c_2-b_{22})\right)k^4 \\ &\quad + 2(c_1-b_{11})(c_2-b_{22})((c_2-b_{22})d_1 + (c_1-b_{11})d_2)k^2 \\ &\quad + (c_1-b_{11})^2(c_2-b_{22})^2 - a_{11}^2 a_{22}^2. \end{aligned}$$

Then, the roots of (3.4) are

$$\omega_k^{\pm} = \frac{-\left(A_k^2 - 2B_k\right) \pm \sqrt{\left(A_k^2 - 2B_k\right)^2 - 4\left(B_k^2 - C^2\right)}}{2}. \tag{3.5}$$

Thus, if

$$\text{(H4)} \begin{cases} A_k^2 - 2B_k > 0 \text{ and } B_k^2 - C^2 > 0, \\ \left(A_k^2 - 2B_k\right)^2 < 4\left(B_k^2 - C^2\right), \end{cases}$$

holds for any $k \in K_0$, then (3.4) has no positive root. Therefore, (2.3) has no $\pm i\omega$ roots. Since (H1)–(H3) ensure system, (1.1) is locally asymptotically stable at zero equilibrium when $\tau = 0$, and following the Corollary 2.4 in [19], we have:

Lemma 1 Suppose (H1)–(H3) holds

1. *If* (H4) *holds for arbitrary* $k \in K_0$, *then all the roots of* (2.3) *have negative real parts for any* $\tau \geq 0$.
2. *If there exists* $k \in K_0$ *such that*
   (H5) $B_k^2 - C^2 < 0$, *then* (3.4) *has one positive root* ω_k^+.
3. *If there exists* $k \in K_0$ *such that*
   (H6) $A_k^2 - 2B_k < 0, B_k^2 - C^2 > 0$ *and* $\left(A_k^2 - 2B_k\right)^2 \geq 4\left(B_k^2 - C^2\right)$.
   then (3.4) *has two positive roots* $\omega_k^{\pm}$.

For the last two cases, (2.3) has purely imaginary roots as long as τ takes some critical values $\tau_{k,j}^+\left(\tau_{k,j}^-\right)$, which is as follows:

$$\tau_{k,j}^{\pm} = \frac{1}{2\omega_k^{\pm}}\left(\arccos\left(\frac{\left(\omega_k^{\pm}\right)^2 - B_k}{C}\right) + 2j\pi\right) \quad j, k \geq 0. \tag{3.6}$$

Now, we only analyze the case of $k = 0$ to illustrate the existence of positive root of (3.4). When $k = 0$, (3.4) becomes

$$\omega^4 + \left(A_0^2 - 2B_0\right)\omega^2 + B_0^2 - C^2 = 0, \tag{3.7}$$

where

$$\begin{aligned} A_0^2 - 2B_0 &= (c_1-b_{11})^2 + (c_2-b_{22})^2, \\ B_0^2 - C^2 &= (c_1-b_{11})^2(c_2-b_{22})^2 - a_{11}^2 a_{22}^2. \end{aligned}$$

Obviously, it is easy to see $A_0^2 - 2B_0 > 0$. From the (H1) and (H2), we have $(c_1 - b_{11})^2 + a_{11}a_{22} < 0, c_1 > b_{11}$, and $c_2 > c_0$, which can deduce that ${B_0}^2 - C^2 < 0$; combining with $A_0^2 - 2B_0 > 0$, one can obtain that (3.7) has at least one positive root. Besides, from (H3), we have $c_2 > b_{22}$, $c_1 > b_{11}$; thus, if $(c_1 - b_{11})(c_2 - b_{22}) + a_{11}a_{22} < 0$ is satisfied, we can obtain ${B_0}^2 - C^2 < 0$. If (3.7) has at least one positive root, the positive root can be defined as ω_0. On the other hand, if (H3) and $(c_1 - b_{11})(c_2 - b_{22}) + a_{11}a_{22} > 0$ hold, one can obtain ${B_0}^2 - C^2 > 0$, which means (3.7) has no positive root.

Corollary 1 (1) *If* (H1)–(H2) *hold*, *then*, *there exists* $k \in K_0$ *such that* (3.4) *has at least one positive root.*

(2) If (H3) hold, here exists $k \in K_0$ such that (3.4) has at least one positive root when $(c_1 - b_{11})(c_2 - b_{22}) + a_{11}a_{22} < 0$ is satisfied, and (3.4) has no positive root with $(c_1 - b_{11})(c_2 - b_{22}) + a_{11}a_{22} > 0$.

Lemma 2 *Suppose that* (H1)–(H3) *holds.*

1. *If* (H5) *holds for some* $k \in K_0$, *then* $\mathrm{Re}\lambda'\left(\tau_{k,j}^+\right) > 0, j \geq 0$.

2. *If* (H6) *holds for some* $k\in K_0$, *then for* $j\geq 0$, *we have*

(1) $\mathrm{Re}\lambda'\left(\tau_{k,j}^{\pm}\right)=0$ *when* $\left(A_k^2-2B_k\right)^2-4\left(B_k^2-C^2\right)=0$.

(2) $\mathrm{Re}\lambda'\left(\tau_{k,j}^{+}\right)>0$ 、 $\mathrm{Re}\lambda'\left(\tau_{k,j}^{-}\right)<0$ *when*

$\left(A_k^2-2B_k\right)^2-4\left(B_k^2-C^2\right)>0$.

Proof From (2.3), we have

$$(2\lambda+A_k)\frac{d\lambda}{d\tau}+C\left(-2\tau\frac{d\lambda}{d\tau}-2\lambda\right)e^{-2\lambda\tau}=0. \tag{3.8}$$

Then,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{(2\lambda+A_k)e^{2\lambda\tau}}{2\lambda C}-\frac{\tau}{\lambda}. \tag{3.9}$$

By (3.9), one can obtain

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_{k,j}^{\pm}}=\frac{\pm\sqrt{\left(A_k^2-2B_k\right)^2-4\left(B_k^2-C^2\right)}}{2C^2}, \tag{3.10}$$

As $\left(A_k^2-2B_k\right)^2-4\left(B_k^2-C^2\right)>0$, one can obtain $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\Big|_{\tau=\tau_{k,j}^{\pm}}\neq 0$. The proof is complete.

Denote $A_1=\{k\geq 0:(H5)\ holds\}$, $A_2=\{k\geq 0:(H6)\ holds\}$. It is obvious from (3.3) that $\left\{\tau_{k,j}^{\pm}\right\}\Big|_{j=0}^{\infty}$ is increasing on j for the fixed $k\in A_2$. So, for the fixed k, we have $\tau_{k,0}^{\pm}=\min_{j\geq 0}\left\{\tau_{k,j}^{\pm}\right\}$. For all $k\in A_1\cup A_2$, we define the smallest critical value, which is as follows:

$$\tau_0\overset{def}{=}\begin{cases}\min\left\{\tau_{k,0}^{+}\right\},\\ \min\left\{\tau_{k,0}^{+},\tau_{k,0}^{-}\right\}.\end{cases}$$

From (3.3), as $y=\arccos(\cdot)$ is a decreasing function, thus, for $\omega_k^-<\omega_k^+$, we can obtain $\tau_k^->\tau_k^+$ for any $k\in A_2$ when (H6) holds. Define $\tau_0=\tau_{k,0}^{+}=\min\left\{\tau_{k,0}^{+}:k\in A_2\right\}$, by [19], we can obtain that all the roots of (2.3) have negative real parts when $\tau\in[0,\tau_0)$. For $\tau=\tau_{k,j}^{+}\left(\tau=\tau_{k,j}^{-}\right)$, the (2.3) has a pair of purely imaginary roots $\pm i\omega_k^+\left(\pm i\omega_k^-\right)$.

From lemmas 1–2 and corollary 1, the following conclusion can be obtained.

Theorem 2 (1) *If* (H1) *or* (H2) *hold, then the system* (1.1) *is locally asymptotically stable at zero equilibrium when* $\tau\in[0,\tau_0)$; *and system* (1.1) *undergoes a Hopf bifurcation at the origin when* $\tau=\tau_0$.

(2) *If* (H3) holds,

1) *if* $(c_1-b_{11})(c_2-b_{22})+a_{11}a_{22}<0$, *then the system* (1.1) *is locally asymptotically stable at zero equilibrium when* $\tau\in[0,\tau_0)$; *and system* (1.1) *undergoes a Hopf bifurcation at the origin when* $\tau=\tau_0$.

2) *if* $(c_1-b_{11})(c_2-b_{22})+a_{11}a_{22}>0$, *then the system* (1.1) *is locally asymptotically stable at zero equilibrium for any* $\tau\geq 0$.

Theorem 3 *Suppose* $(b_{11}-c_1)^2+a_{11}a_{22}<0,\ c_2>c_0$, $\tau\in[0,\tau_0)$ *hold*, then we have:

(1) *If* $c_2>b_{22},\ c_1>b_{11},\ (c_1-b_{11})(c_2-b_{22})+a_{11}a_{22}<0$, *the system* (1.1) *is locally stable at zero equilibrium*;

(2) *If* $c_2>b_{22},\ c_1<b_{11}$*and* $d_1/d_2\in(\gamma_1,+\infty)$, *then the system* (1.1) *is locally asymptotically stable at zero equilibrium*;

(3) *If* $c_2>b_{22},\ c_1<b_{11}$*and* $d_1/d_2\in(0,\gamma_1)$, *then the system* (1.1) *without diffusion is local stable, and become Turing unstable with diffusion*.

(4) *If* $c_2<b_{22},\ c_1>b_{11}$*and, then the system* (1.1) *is locally asymptotically stable*;

(5) *If* $c_2<b_{22},\ c_1>b_{11}$*and* $d_1/d_2\in(\gamma_2,+\infty)$, *then the system* (1.1) *without diffusion is local stable, and become Turing unstable with diffusion*.

Remark 2 As is known to all, the Turing instability is one of the famous phenomenon in the reaction-diffusion system. However, very few works investigate the Turing bifurcation of RDNNS. In this paper, we give the conditions of the Turing bifurcation of RDNNs with and without time delay.

Remark 3 In [14], the authors give the Hopf conditions of system (1.1) which are $c_2>b_{22},\ c_1>b_{11}$ and $(c_1-b_{11})(c_2-b_{22})+a_{11}a_{22}<0$. From Theorem 2, it is easy to see that the Hopf conditions of [14] are also a subset of our results.

Remark 4 As known to all, the diffusion coefficients play a key role on the dynamic behaviors of RDNN such as stability and bifurcations. However, the condition of stability and Hopf bifurcation of the existing works [14–16] does not contain the diffusion coefficient. In this paper, our results show the diffusion can lead the dynamic behaviors of RDNNs change.

The Normal Form of Hopf Bifurcation

Let $\vartheta=\tau-\tau_0$ and $t\rightarrow t/\tau$, (1.1) can be rewritten as:

$$\dot{R}(t)=\tau_0DR(t)+L(\tau_0)(R(t))+F(R(t),\vartheta) \tag{4.1}$$

where

$D = diag(d_1, d_2)$,

$L(\tau_0)(\varphi) = \tau_0(B_0\varphi(0) + B_1\varphi(-\tau))$,

$B_0 = \begin{bmatrix} -c_1 + b_{11} & 0 \\ 0 & -c_2 + b_{22} \end{bmatrix} B_1 = \begin{bmatrix} 0 & a_{11} \\ a_{22} & 0 \end{bmatrix}$.

$F(\varphi, \vartheta) = \vartheta D\Delta\varphi(0) + L(\vartheta)(\varphi) + f^*(\varphi, \vartheta)$,

$f^*(\varphi, \vartheta) = (\tau_0 + \vartheta)\begin{pmatrix} l_{11}\varphi_1^2(0) + l_{12}\varphi_1^3(0) + m_{11}\varphi_2^2(-1) + m_{12}\varphi_2^3(-1)\ldots \\ l_{21}\varphi_2^2(0) + l_{22}\varphi_2^3(0) + m_{21}\varphi_1^2(-1) + m_{22}\varphi_1^3(-1)\ldots \end{pmatrix}$,

For $\varphi = (\varphi_1, \varphi_2)^T \in \zeta$,

$l_{11} = \frac{b_1 f_1''(0)}{2!}, l_{12} = \frac{b_1 f_1'''(0)}{3!}, l_{21} = \frac{b_2 f_3''(0)}{2!}, l_{22} = \frac{b_2 f_3'''(0)}{3!}$,

$m_{11} = \frac{a_1 f_2''(0)}{2!}, m_{12} = \frac{a_1 f_2'''(0)}{3!}, m_{21} = \frac{a_2 f_4''(0)}{2!}, m_{22} = \frac{a_2 f_4'''(0)}{3!}$.

It is not hard to see that $\Lambda_0(-i\omega_0\tau_0, i\omega_0\tau_0)$ are the eigenvalues of the linear part of (4.1):

$$\dot{R}(t) = \tau_0 DR(t) + L(\tau_0)(R(t)) \quad (4.2)$$

and

$$\dot{z}(t) = L(\tau_0)(z_t) \quad (4.3)$$

where $L(\tau_0)$ is linear operator in $C \coloneqq C([-1, 0]\mathfrak{R}^2)$ into $\mathfrak{R}^2$.

By [24–26], we define $\eta(\theta, \tau)$ for $\theta \in [-1, 0]$, such that

$$L(\tau_0)(\varphi) = \int_{-1}^0 d\eta(\theta, \tau_0)\varphi(\theta), \varphi \in C \quad (4.4)$$

where

$\eta(\theta, \tau_0) = (\tau_0 + \vartheta)[B_0\delta(\theta) + B_1\delta(\theta + 1)]$,

$\delta(\theta)$ is Drac-delta function.

Let $A(\tau_0)$ be the infinitesimal generator of the semigroup induce by the solution of (4.3) and A^* denotes the formal adjoint of $A(\tau_0)$ under the bilinear pairing

$$\langle\psi, \varphi\rangle = \psi(0)\cdot\varphi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \psi(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \quad (4.5)$$

for $\phi \in C$, $\psi \in C^* = C([0, 1], \mathbb{R}^2)$.

It is easy to see that $i\tau_0\omega_0$ is an eigenvalue of $A(\tau_0)$ and $-i\tau_0\omega_0$ is an eigenvalue of A^*. Define $q_1(\theta) = (1, \alpha_1)^T e^{i\omega_0\tau_0\theta}$ and $q_2(\theta) = \overline{q_1}(\theta)$, then one can obtain:

$$Aq_1(0) = i\omega_0\tau_0 q_1(0) \quad (4.6)$$

By calculation, we have

$\alpha_1 = \alpha_{11} + i\alpha_{12}$,

where

$\alpha_{11} = \frac{(c_1 - b_{11})\cos\omega_0\tau_0 + \omega_0\tau_0\sin\omega_0\tau_0}{a_{11}}, \alpha_{12} = \frac{\omega_0\tau_0\cos\omega_0\tau_0 - (c_1 - b_{11})\sin\omega_0\tau_0}{a_{11}}$

Define $p_1(\theta^*) = (1, \beta_1)^T e^{-i\omega_0\tau_0\theta^*}$ and $p_2(\theta^*) = \overline{p}_1(\theta^*)$ is a basis of P^* associated with ζ_0, then we have:

$$A^* p_1(0) = -i\omega_0\tau_0 p_1(0) \quad (4.7)$$

By calculation, we have

$\beta_1 = \beta_{11} + i\beta_{12}$,

where

$\beta_{11} = \frac{a_{11}((c_2 - b_{22})\cos\omega_0\tau_0 + \omega_0\tau_0\sin\omega_0\tau_0)}{(c_2 - b_{22})^2 + \omega_0^2\tau_0^2}$,

$\beta_{12} = \frac{a_{11}(\omega_0\tau_0\cos\omega_0\tau_0 - (c_2 - b_{22})\sin\omega_0\tau_0)}{(c_2 - b_{22})^2 + \omega_0^2\tau_0^2}$.

Let $u(\theta) = (u_1(\theta), u_2(\theta))$ and $v^*(\theta^*) = (v_1^*(\theta^*), v_2^*(\theta^*))^T$ with

$$u_1(\theta) = \frac{q_1(\theta) + q_2(\theta)}{2} = (\mathrm{Re}\{e^{i\omega_0\tau_0\theta}\}, \mathrm{Re}\{\alpha_1 e^{i\omega_0\tau_0\theta}\}) = \begin{pmatrix} \cos\omega_0\tau_0\theta \\ \alpha_{11}\cos\omega_0\tau_0\theta - \alpha_{12}\sin\omega_0\tau_0\theta \end{pmatrix},$$

$$u_2(\theta) = \frac{q_1(\theta) - q_2(\theta)}{2i} = (\mathrm{Im}\{e^{i\omega_0\tau_0\theta}\}, \mathrm{Im}\{\alpha_1 e^{i\omega_0\tau_0\theta}\}) = \begin{pmatrix} \sin\omega_0\tau_0\theta \\ \alpha_{11}\sin\omega_0\tau_0\theta + \alpha_{12}\cos\omega_0\tau_0\theta \end{pmatrix},$$

for $\theta \in [-1, 0]$ and

$$v_1^*(\theta^*) = \frac{p_1(\theta^*) + p_2(\theta^*)}{2} = \left(\mathrm{Re}\left\{e^{-i\omega_0\tau_0\theta^*}\right\}, \mathrm{Re}\left\{\beta_1 e^{-i\omega_0\tau_0\theta^*}\right\}\right) = \begin{pmatrix} \cos\omega_0\tau_0\theta^* \\ \beta_{11}\cos\omega_0\tau_0\theta^* + \beta_{12}\sin\omega_0\tau_0\theta^* \end{pmatrix},$$

$$v_2(\theta^*) = \frac{p_1(\theta^*) - p_2(\theta^*)}{2i} = \left(Im\left\{e^{-i\omega_0\tau_0\theta^*}\right\}, Im\left\{\beta_1 e^{-i\omega_0\tau_0\theta^*}\right\}\right) = \begin{pmatrix} \sin\omega_0\tau_0\theta^* \\ -\beta_{11}\sin\omega_0\tau_0\theta^* + \beta_{12}\cos\omega_0\tau_0\theta^* \end{pmatrix},$$

for $\theta^* \in [-1, 0]$.

In the following, define $(u,v^{*})=\left(u_j,v_k^{*}\right)j,k=1,2$, then we have

$$v=(v_1,v_2)^T=\left(v^{*},u\right)^{-1}v^{*}.$$

Let $f_k=\left(\beta_k^1\ \ \beta_k^2\right)$ and $c{\cdot}f_0=c_1\beta_0^1+c_1\beta_0^2$ for $c=(c_1,c_2)^T\in\zeta$, one can obtain:

$$P_{CN}\zeta^{*}=u(v,<\varphi,f_k>){\cdot}f_k\ \ \varphi\in\zeta^{*} \tag{4.8}$$

and $\zeta^{*}=P_{CN}\zeta^{*}\oplus Q$, where Q is the complement subspace of $P_{CN}\zeta^{*}$ in ζ^{*}.

According to the definition of $A(\tau_0)$, we can rewrite (4.1) as

$$\dot{R}(t)=A_{\tau_0}R(t)+X_0F(U_t,\vartheta) \tag{4.9}$$

where

$$X_0(\theta)=\begin{cases}0, & -1\le\theta<0\\ I & \theta=0\end{cases}.$$

By using the decompositions $\zeta^{*}=P_{CN}\zeta^{*}\oplus Q$ and (4.8), (4.2) can be written as

$$R(t)=u\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}{\cdot}f_k+h(x_1,x_2,\vartheta) \tag{4.10}$$

where $(x_1(t),x_2(t))^T=(v,\langle R(t),f_k\rangle)$ and $h(x_1,x_2,\vartheta)\in Q$, $h(0,0,0)=Dh(0,0,0)=0$. Using center manifold, one can obtain:

$$R(t)=u\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}{\cdot}f_k+h(x_1,x_2,0). \tag{4.11}$$

Let $z=x_1-ix_2$ and $q_1=u_1+iu_2$, from (34), we can obtain:

$$R(t)=\frac{1}{2}\left(q_1z+\overline{q}_1\overline{z}\right){\cdot}f_k+W\left(z,\overline{z}\right) \tag{4.12}$$

where

$$W\left(z,\overline{z}\right)=h\left(\frac{z+\overline{z}}{2},\frac{i\left(z-\overline{z}\right)}{2},0\right)\triangleq W_{20}\frac{z^2}{2}+W_{11}z\overline{z}+W_{02}\frac{\overline{z}^2}{2}+\dots \tag{4.13}$$

From [24–26], we have

$$\dot{z}=i\omega_0\tau_0z+g\left(z,\overline{z}\right), \tag{4.14}$$

where

$$g\left(z,\overline{z}\right)=(v_1-iv_2)\langle f(R(t),0),f_k\rangle\triangleq g_{20}\frac{z^2}{2}+g_{11}z\overline{z}+g_{20}\frac{\overline{z}^2}{2}+\dots \tag{4.15}$$

for $v(0)=(v_1(0),v_2(0))^T$.

By calculation, from (4.15), we have

$$\langle f(R(t),0),f_k\rangle=\frac{\tau_0}{4}\frac{1}{\pi}\int_0^{\pi}\cos^3kxdx\begin{pmatrix}d_{11}\\d_{21}\end{pmatrix}\frac{z^2}{2}+\frac{\tau_0}{4}\frac{1}{\pi}\int_0^{\pi}\cos^3kxdx\begin{pmatrix}d_{12}\\d_{22}\end{pmatrix}z\overline{z}+\frac{\tau_0}{4}\frac{1}{\pi}\int_0^{\pi}\cos^3kxdx\begin{pmatrix}d_{13}\\d_{23}\end{pmatrix}\frac{\overline{z}^2}{2}+\frac{\tau_0}{2}\begin{pmatrix}d_{14}\\d_{24}\end{pmatrix}\frac{z^2\overline{z}}{2}.$$

where

$$d_{11}=2l_{11}+2m_{11}\alpha_1^2e^{-2i\tau_0\omega_0},d_{12}=2l_{11}+2m_{11}\alpha_1\overline{\alpha}_1,d_{13}=2l_{11}+2m_{11}\overline{\alpha}_1^2e^{2i\tau_0\omega_0},$$

$$d_{14}=2\left\langle\left(2l_{11}W_{11}+l_{11}W_{20}+2m_{11}\alpha_1W_{11}e^{-i\tau_0\omega_0}+m_{11}\overline{\alpha}_1W_{20}e^{i\tau_0\omega_0}\right)\cos kx,\cos kx\right\rangle+\left(\frac{3}{2}m_{12}\alpha_1^2\overline{\alpha}_1e^{-i\tau_0\omega_0}+\frac{3}{2}l_{12}\right)\frac{1}{\pi}\int_0^{\pi}\cos^4kxdx$$

$$d_{21}=2l_{21}\alpha_1^2+2m_{21}\alpha_1^2e^{-2i\tau_0\omega_0},d_{22}=2l_{21}\alpha_1\overline{\alpha}_1+2m_{21}\alpha_1\overline{\alpha}_1,d_{23}=2l_{21}\overline{\alpha}_1^2+2m_{21}\overline{\alpha}_1^2e^{2i\tau_0\omega_0},$$

$$d_{24}=2\left\langle\left(2l_{21}\alpha_1W_{11}+l_{21}\overline{\alpha}_1W_{20}+2m_{21}\alpha_1W_{11}e^{-i\tau_0\omega_0}+m_{21}\overline{\alpha}_1W_{20}e^{i\tau_0\omega_0}\right)\cos kx,\cos kx\right\rangle+\left(l_{22}\frac{3}{2}\alpha_1^2\overline{\alpha}_1+\frac{3}{2}m_{22}e^{-i\tau_0\omega_0}\right)\frac{1}{\pi}\int_0^{\pi}\cos^4kxdx$$

$$\left\langle W_{ij}^n(\theta),\cos kx\right\rangle=\frac{1}{\pi}\int_0^{\pi}W_{ij}^n(\theta)(x)dx,\ \ i+j=2,n=1,2,3,4.$$

Notice that $\frac{1}{\pi}\int_0^{\pi}\cos^3kxdx=0$ when $k\neq0$. Let $(\rho_1,\rho_2)=v_1(0)-iv_2(0)$, compare the coefficients with (4.15), we can obtain

$$g_{20}=\begin{cases}0, & k\in K\\ \frac{\tau_0}{4}(d_{11}\rho_1+d_{21}\rho_2), & k=0\end{cases},\quad g_{11}=\begin{cases}0, & k\in K\\ \frac{\tau_0}{4}(d_{12}\rho_1+d_{22}\rho_2), & k=0\end{cases},\quad g_{02}=\begin{cases}0, & k\in K\\ \frac{\tau_0}{4}(d_{13}\rho_1+d_{23}\rho_2), & k=0\end{cases},g_{21}=\frac{\tau_0}{2}(d_{14}\rho_1+d_{24}\rho_2). \tag{4.16}$$

Since W_{11} and W_{20} in g_{21} for $\theta\in[-1,0]$ are uncertainty, we need to further determine them. It follows from (4.13) that

$$\dot{W}\left(z,\overline{z}\right)=W_{20}z\dot{z}+W_{11}\dot{z}\overline{z}+W_{11}z\dot{\overline{z}}+W_{20}\overline{z}\dot{\overline{z}}+\cdots, \tag{4.17}$$

And

$$A_{\tau_0}W = A_{\tau_0}W_{20}\frac{z^2}{2} + A_{\tau_0}W_{11}z\overline{z} + A_{\tau_0}W_{02}\frac{\overline{z}^2}{2} + \ldots, \quad (4.18)$$

By using center manifold, we have

$$\dot{W} = A_{\tau_0}W + H\left(z, \overline{z}\right), \quad (4.19)$$

where

$$H\left(z, \overline{z}\right) = H_{20}\frac{z^2}{2} + H_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots$$
$$= X_0 f^*(U_t, 0) - \Phi\left(\Psi, \langle X_0 f^*(U_t, 0), f_k\rangle \cdot f_k\right). \quad (4.20)$$

We derive from (4.14), (4.19), and (4.20) that

$$(A_{\tau_0} - 2i\tau_0\omega_0)W_{20}(\theta) = -H_{20}(\theta), \cdots A_{\tau_0}W_{11}(\theta) = -H_{11}(\theta). \quad (4.21)$$

By a straightforward calculation and then by (4.20), we obtain that

$$\begin{aligned} H\left(z, \overline{z}\right) &= X_0 f^*(R(t), 0) - u\left(v, \langle X_0 f^*(R(t), 0), f_k\rangle \cdot f_k\right) \\ &= -\left(\frac{q_1(\theta) + q_2(\theta)}{2}, \frac{q_1(\theta) - q_2(\theta)}{2i}\right)(v_1(0), v_2(0))\langle f^*(R(t), 0), f_k\rangle \cdot f_k \\ &= -\frac{1}{2}q_1(\theta)(v_1(0) - iv_2(0))\langle f^*(R(t), 0), f_k\rangle \cdot f_k - \frac{1}{2}q_2(\theta)(v_1(0) + iv_2(0))\langle f^*(R(t), 0), f_k\rangle \cdot f_k \\ &= -\frac{1}{2}\left[q_1(\theta)g_{20} + q_2(\theta)\overline{g}_{02}\right] \cdot f_k \frac{z^2}{2} - \frac{1}{2}\left[q_1(\theta)g_{11} + q_2(\theta)\overline{g}_{11}\right] \cdot f_k z\overline{z}. \end{aligned}$$

Then, we have

$$H_{20} = \begin{cases} 0, & k \in K, \\ -\frac{1}{2}\left[q_1(\theta)g_{20} + q_2(\theta)\overline{g}_{02}\right] \cdot f_0, & k = 0, \end{cases} \quad (4.22)$$

and

$$H_{11} = \begin{cases} 0, & k \in K, \\ -\frac{1}{2}\left[q_1(\theta)g_{11} + q_2(\theta)\overline{g}_{11}\right] \cdot f_0, & k = 0. \end{cases} \quad (4.23)$$

From (4.21), (4.22), and the definition of A_{τ_0}, we have

$$\dot{W}_{20}(\theta) = 2i\tau_0\omega_0 W_{20}(\theta) + \frac{1}{2}\left[q_1(\theta)g_{20} + q_2(\theta)\overline{g}_{02}\right] \cdot f_k. \quad (4.24)$$

Notice that $q_1(\theta) = q_1(0)e^{i\omega_0\tau_0}$, we therefore obtain that

$$W_{20}(\theta) = \frac{1}{2}\left[\frac{ig_{20}}{\tau_0\omega_0}q_1(\theta) + \frac{i\overline{g}_{20}}{3\tau_0\omega_0}q_2(\theta)\right] \cdot f_k + E_1 e^{2i\tau_0\omega_0\theta}. \quad (4.25)$$

where $E_1 = \left(E_1^{(1)}, E_1^{(2)}\right) \in R^2$.

From (4.21) and (4.23), we have

$$W_{11}(\theta) = \frac{1}{2}\left[\frac{-ig_{11}}{\tau_0\omega_0}q_1(\theta) + \frac{i\overline{g}_{11}}{\tau_0\omega_0}q_2(\theta)\right] \cdot f_k + E_2. \quad (4.26)$$

where $E_2 = \left(E_2^{(1)}, E_2^{(2)}\right) \in R^2$ is also a constant vector.

From (4.21), one can obtain:

$$\begin{aligned} H_{20}(0) &= 2i\tau_0\omega_0 W_{20}(0) - \tau_0 D\Delta W_{20}(0) - L(\tau_0)W_{20}(\theta), \\ H_{11}(0) &= -\tau_0 D\Delta W_{11}(0) - L(\tau_0)W_{11}(\theta), \end{aligned} \quad (4.27)$$

where

$$H_{20}(0) = \begin{cases} \frac{\tau_0}{4}(d_{11}, d_{21})^T\cos^2 kx, & k \in K, \\ \frac{\tau_0}{4}(d_{11}, d_{21})^T - \frac{1}{2}\left[q_1(0)g_{20} + q_2(0)\overline{g}_{02}\right] \cdot f_0, & k = 0, \end{cases}$$

$$H_{11}(0) = \begin{cases} \frac{\tau_0}{4}(d_{11}, d_{21})^T\cos^2 kx, & k \in K, \\ \frac{\tau_0}{4}(d_{11}, d_{21})^T - \frac{1}{2}\left[q_1(0)g_{11} + q_2(0)\overline{g}_{11}\right] \cdot f_0, & k = 0. \end{cases}$$

We derive from (4.25) and (4.27) that

$$
\begin{aligned}
&2i\tau_0\omega_0\left[\frac{1}{2}\left[\frac{ig_{20}}{\tau_0\omega_0}q_1(0)+\frac{i\overline{g}_{20}}{3\tau_0\omega_0}q_2(0)\right]\cdot f_0+E_1\right]\\
&-\tau_0 D\Delta\left[\frac{1}{2}\left[\frac{ig_{20}}{\tau_0\omega_0}q_1(0)+\frac{i\overline{g}_{20}}{3\tau_0\omega_0}q_2(0)\right]\cdot f_0+E_1\right]\\
&-L(\tau_0)\left[\frac{1}{2}\left[\frac{ig_{20}}{\tau_0\omega_0}q_1(\theta)+\frac{i\overline{g}_{20}}{3\tau_0\omega_0}q_2(\theta)\right]\cdot f_0+E_1e^{2i\tau_0\omega_0\theta}\right]\\
&=\frac{\tau_0}{4}(d_{11},d_{21})^T-\frac{1}{2}\left[q_1(0)g_{20}+q_2(0)\overline{g}_{02}\right]\cdot f_0.
\end{aligned}
\tag{4.28}
$$

Note that

$$
\begin{aligned}
&\tau_0 D\Delta[q_1(0)\cdot f_0]+L(\tau_0)[q_1(\theta)\cdot f_0]=i\tau_0\omega_0 q_1(0)\cdot f_0,\\
&\tau_0 D\Delta[q_2(0)\cdot f_0]+L(\tau_0)[q_2(\theta)\cdot f_0]=-i\tau_0\omega_0 q_2(0)\cdot f_0.
\end{aligned}
$$

Then for $k\in K_0$, it follows from (4.28) that

$$
E_1=\frac{1}{4}\begin{pmatrix}2i\omega_0+d_1k^2+c_1-b_{11} & -a_{11}e^{-2i\omega_0\tau_0}\\ -a_{22}e^{-2i\omega_0\tau_0} & 2i\omega_0+d_2k^2+c_2-b_{22}\end{pmatrix}^{-1}\begin{pmatrix}d_{11}\\ d_{21}\end{pmatrix}\cos^2 kx=0,
\tag{4.29}
$$

By a similar analysis, we obtain

$$
E_2=\frac{1}{4}\begin{pmatrix}d_1k^2+c_1-b_{11} & -a_{11}\\ -a_{22} & d_2k^2+c_2-b_{22}\end{pmatrix}^{-1}\begin{pmatrix}d_{11}\\ d_{21}\end{pmatrix}\cos^2 kx
$$

$$
=0.
\tag{4.29}
$$

Thus, by (4.25) and (4.26), we can obtain $W_{20}(\theta)$ and $W_{11}(\theta)$. Hence, g_{21} can be expressed. Therefore, the following important parameters can be obtained:

$$
\begin{aligned}
&C_1(0)=\frac{g_{21}}{2},\\
&\mu_2=-\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_0)\}},\\
&\beta_2=2\mathrm{Re}\{C_1(0)\},\\
&T_2=-\frac{\mathrm{Im}\{C_1(0)\}+\mu_2\mathrm{Im}\{\lambda'(\tau_0)\}}{\omega_0\tau_0}.
\end{aligned}
\tag{4.30}
$$

Theorem 4 (1) *The sign(μ_2) can determine the direction of Hopf bifurcation: if $\mu_2>0(\mu_2<0)$, the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exists for $\tau>\tau_0(\tau<\tau_0)$;*

(2) *The sign(β_2) determines the stability of the bifurcating periodic solutions: if $\beta_2<0$, ($\beta_2>0$), the bifurcation periodic solutions are stable(unstable);*

(3) *The sign(T_2) determine the period of the bifurcating periodic solutions: if $T_2>0(T_2<0)$, the period increases (decreases).*

Numerical Examples

In the section, four examples are given to validate the obtained results. Let $f_i(x)=\tanh(x)$, $a_1=-2$, $a_2=2$, $b_1=2$, $b_2=2$, $f_i'(0)=\tan h(0)=1$, and $b_{11}=b_{12}=b_{21}=b_{22}=2$, the initial conditions of system (1.1) are as follows:

$$
\begin{cases}u(t,x)=0.5(1+t/\pi)\sin(0.1*\pi x),\\ v(t,x)=0.5(1+t/\pi)\sin(0.1*\pi x),\end{cases}
$$

with

$$
\partial u/\partial x=\partial v/\partial x=0,\quad t>0, x\in\partial\Omega.
$$

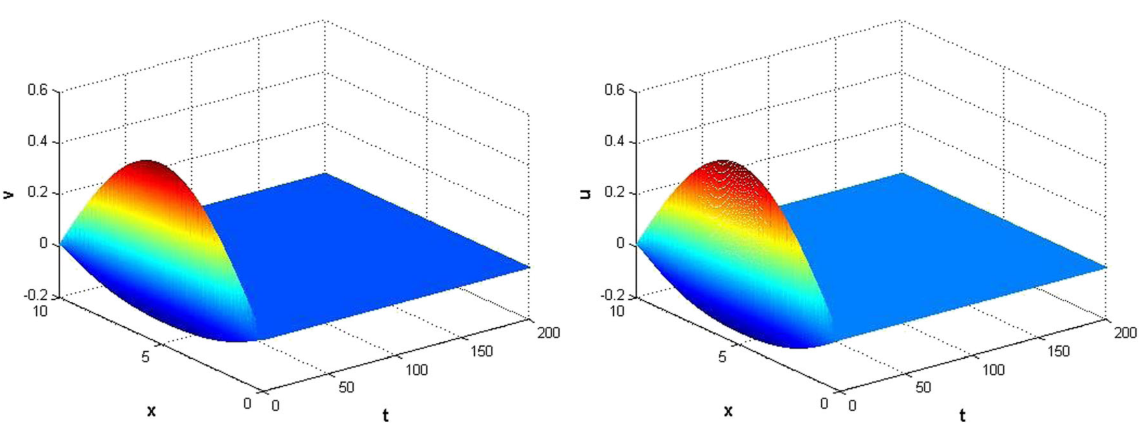

Fig. 1 The equilibrium (0, 0) is stable when $d_1/d_2=0.08>\gamma_1$ at $\tau=0$

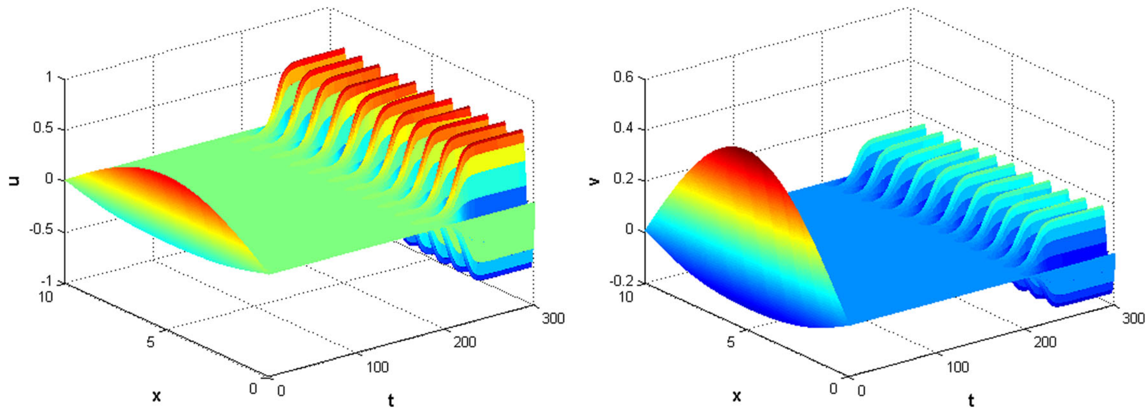

Fig. 2 The equilibrium (0, 0) is unstable when $d_1/d_2 = 0.06 < \gamma_1$ at $\tau = 0$

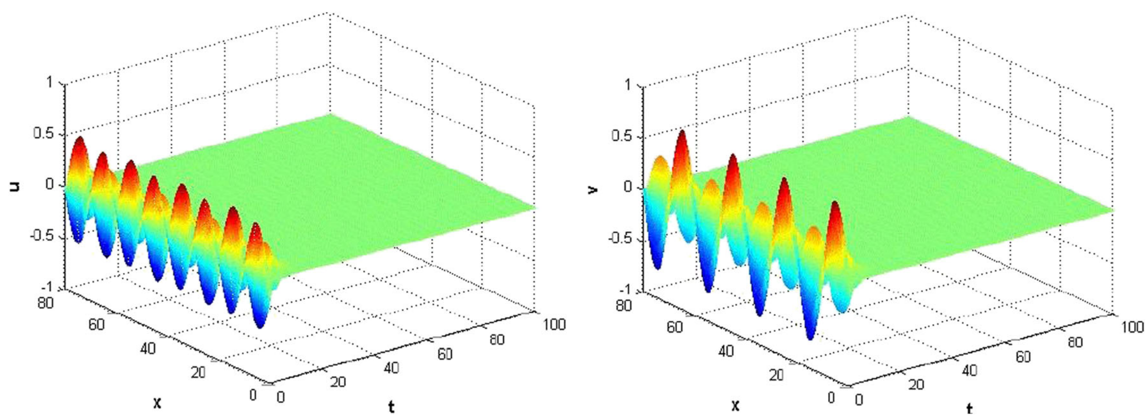

Fig. 3 The equilibrium (0, 0) is stable when $d_1/d_2 = 50 < \gamma_2$ at $\tau = 0$

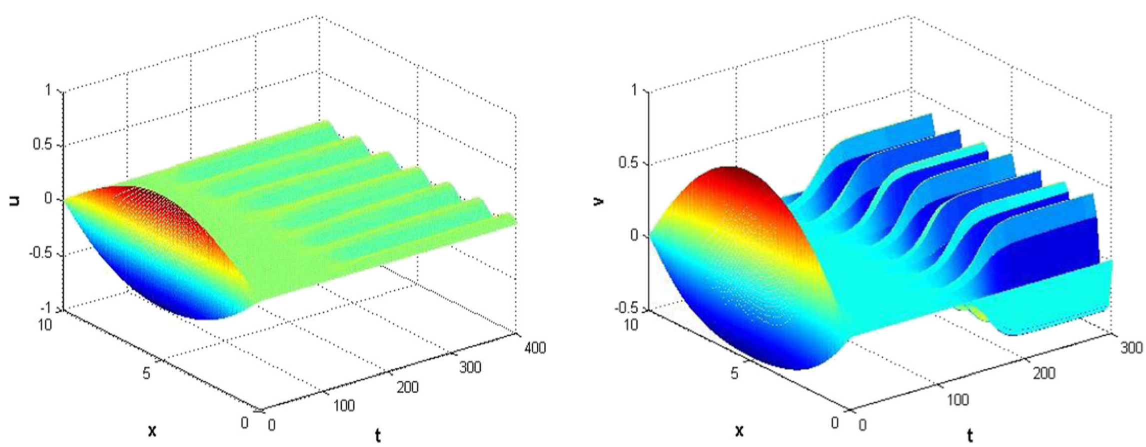

Fig. 4 The equilibrium (0, 0) is unstable when $d_1/d_2 = 70 > \gamma_2$ at $\tau = 0$

Example 1 The example is used to verify Theorem 1. Firstly, let $c_1 = 1$, by calculating, we can obtain $c_0 = 3$. Considering $c_2 = 4 > c_0$, it is easy to see $(b_{11} - c_1)^2 + a_{11}a_{22} < 0$, $c_2 > b_{22}$, $c_1 < b_{11}$, and $c_2 > c_0$ are satisfied. From (2.10) and (2.12), we can obtain $d_{22} = 0.5$, $\gamma_1 = 0.0858$, and $\gamma_2 = 2.9142$. Considering $d_1 = 0.009$, $d_2 = 0.1$ and $d_1/d_2 = 0.09 > \gamma_1$ by Theorem 1, one can obtain system (1.1) is locally asymptotically stable, which is shown in Fig. 1. Considering $d_1 = 0.006$, $d_2 = 0.1$ such that $d_1/d_2 = 0.06 < \gamma_1$ by Theorem 1, one can obtain system (1.1) is Turing unstable, see Fig. 2. The stripe pattern, which is caused by the Turing instable, is shown in Fig. 11a.

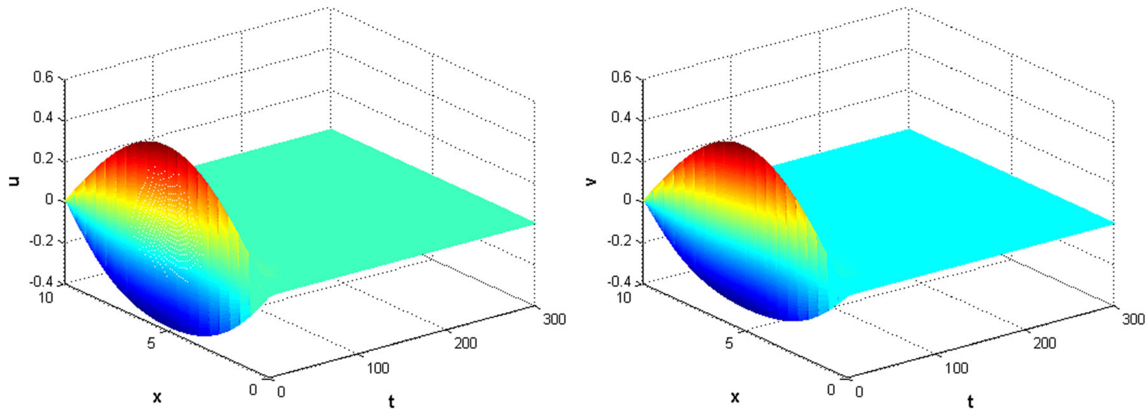

Fig. 5 The equilibrium (0, 0) is stable when $\tau = 0.12 < \tau_0$

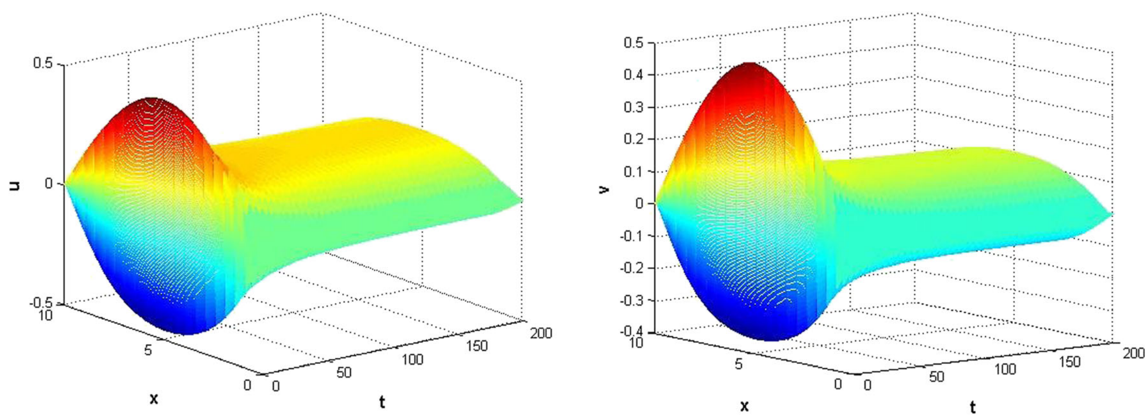

Fig. 6 The equilibrium (0, 0) is unstable when $\tau = 0.13 > \tau_0$

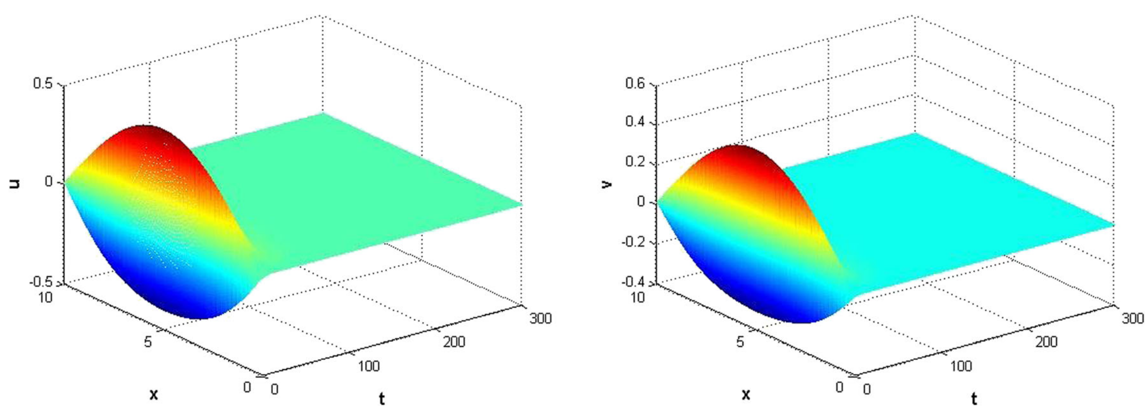

Fig. 7 The equilibrium (0, 0) is stable at $d_1/d_2 = 0.08 > \gamma_1$ and $\tau = 0.11 < \tau_0$

Secondly, let $c_1 = 3$, similarly, we can obtain $c_0 = 1$. Then, considering $c_2 = 1.5 > c_0$ satisfy $(b_{11} - c_1)^2 + a_{11}a_{22} < 0$, $c_2 < b_{22}$, $c_1 > b_{11}$, and $c_2 > c_0$. From (2.10) and (2.12), we can obtain $d_{11} = 2$, $\gamma_1 = 0.0667$, and $\gamma_2 = 59.93$. Considering $d_1 = 0.5$ and $d_2 = 0.01$ such that $d_1/d_2 = 50 < \gamma_2$, by Theorem 1, one can obtain system (1.1) is locally asymptotically stable, see Fig. 3. Considering $d_1 = 0.7$ and $d_2 = 0.01$ such that $d_1/d_2 = 70 > \gamma_2$, by Theorem 1, one can obtain system (1.1) becomes Turing unstable, see Fig. 4. The stripe pattern, which is caused by the Turing instable, is shown in Fig. 11b.

Example 2 The example is used to verify Theorem 2. Firstly, let $c_1 = 1$, $c_2 = 4$, $d_1 = 0.1$, $d_2 = 0.1$, which satisfy the first set of conditions of (H1), and from (3.5) to (3.6), we can obtain ω_0^+

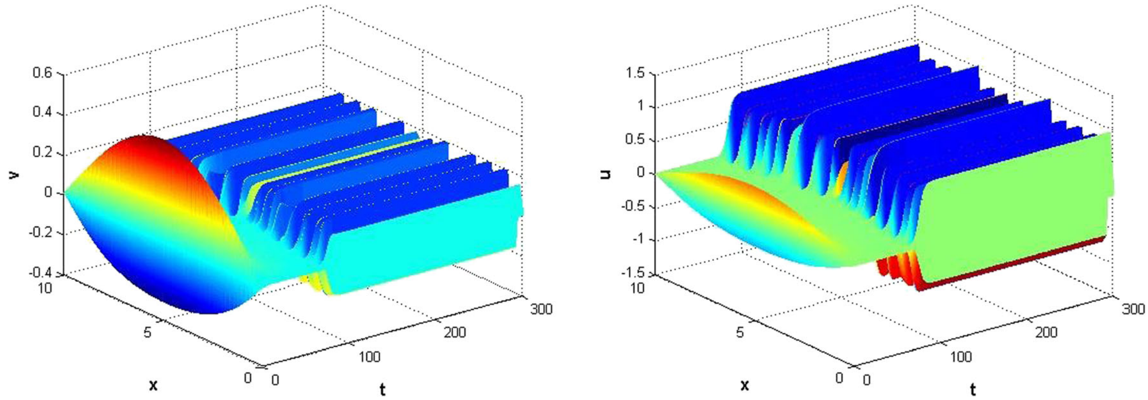

Fig. 8 The equilibrium (0, 0) is unstable at $d_1/d_2 = 0.04 < \gamma_1$ and $\tau = 0.11 < \tau_0$

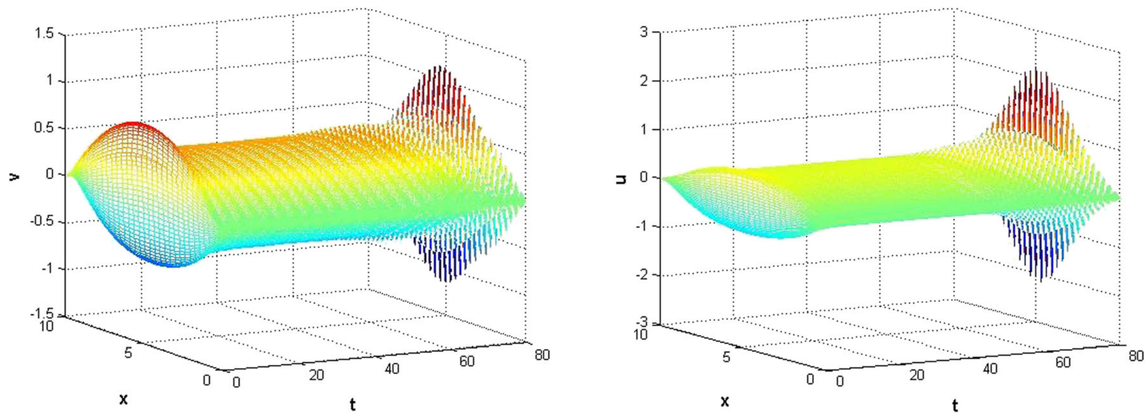

Fig. 9 System (1.1) is unstable at origin with $d_i = 1$

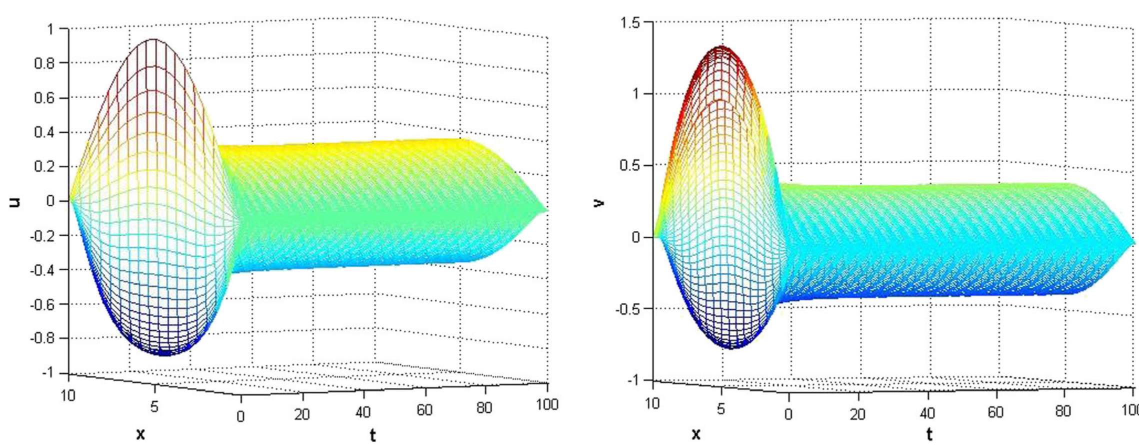

Fig. 10 Periodic solutions of system (1.1) at origin when $d_i = 1.5$

$= 1.331$ and $\tau_0 = 0.127$. Considering $\tau = 0.12 < \tau_0$, by Theorem 2, one can obtain system (1.1) is locally asymptotically stable, see Fig. 5. Consider $\tau = 0.13 > \tau_0$, by Theorem 2, one can obtain that the Hopf bifurcation occurs, which is shown in Fig. 6.

Example 3 The example is used to verify Theorem 3. Considering $\tau = 0.11 < \tau_0$, $d_1 = 0.008$, and $d_2 = 0.1$ such that $d_1/d_2 = 0.08 > \gamma_1$, by Theorem 3, one can obtain system (1.1) is stable, see Fig. 7. Considering $\tau = 0.11 < \tau_0$, $d_1 = 0.004$, and $d_2 = 0.1$ such that $d_1/d_2 = 0.04 < \gamma_1$, by Theorem 1, one can obtain system (1.1) is Turing unstable, see Fig. 8. The stripe pattern, which is cause by the Turing instable, is shown in Fig. 11c.

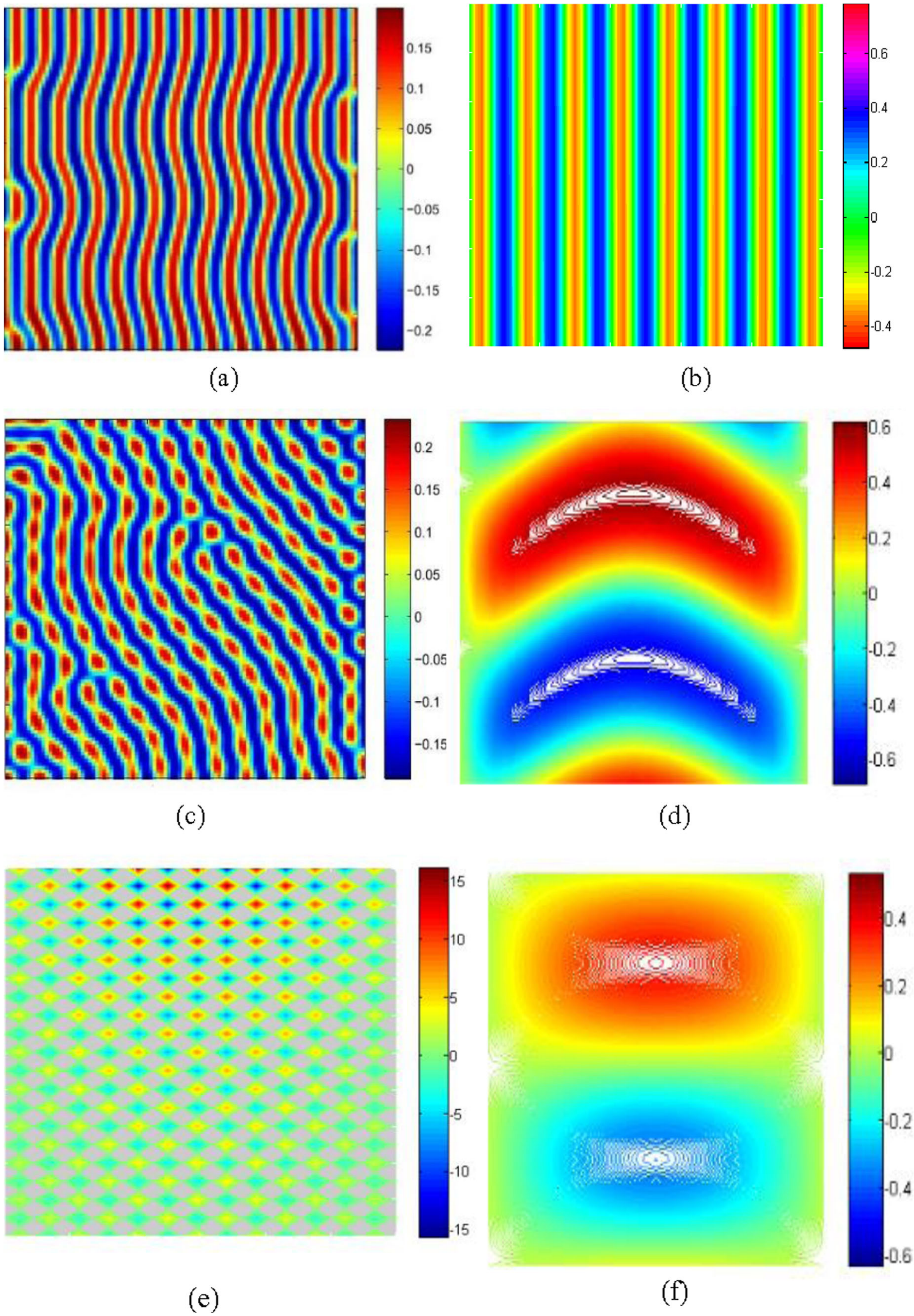

Fig. 11 Spatial pattern appears in system (1.1)

Example 4 This example is used to study the impact of diffusion in the pattern formation of system (1.1). Let $c_1 = 1$, $c_2 = 4$, and $\tau = 0.13 > \tau_0$, we choose $d_i = 0.1, \quad 1, \quad 1.5$, respectively. When $d_i = 0.1$, there exists a stable periodic solution (Fig. 6) and a stripe pattern (Fig. 11d). When $d_i = 1$, the periodic solution becomes unstable (Fig. 9), and there exist irregular patterns (Fig. 11e). When $d_i = 1.5$, the unstable periodic solution becomes stable (Fig. 10) and the stripe patterns (Fig. 11f) are different from the stripe patterns with $d_i = 0.1$.

Conclusion

In this paper, a class of delayed reaction-diffusion neural networks under homogeneous Neumann boundary conditions is considered. First, the conditions of Turing instability of the zero equilibrium are obtained. Then, based on the Turing instability conditions, some sufficient conditions of Hopf bifurcations are obtained. Comparing with the existing works, our results give the novel conditions where the Hopf bifurcation occurs. From the numerical simulation, the spatial temporal dynamic near the Turing instability point can be divided into two types: the stable zero equilibrium and the stripe pattern. The spatial temporal dynamic near the Hopf bifurcation point can be divided into four types: the stable zero equilibrium, the two distinct stripe patterns, and the irregular pattern.

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Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors..

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