



Multiple Attribute Decision-Making with Dual Hesitant Pythagorean Fuzzy Information

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Abstract

Due to the uncertainty and complexity of socioeconomic environments and cognitive diversity of decision makers, the cognitive information over alternatives provided by decision makers is usually uncertain and fuzzy. Dual hesitant Pythagorean fuzzy sets (DHPFSs) provide a useful tool to depict the uncertain and fuzzy cognitions of the decision makers over attributes. To effectively handle such common cases, in this paper, some Bonferroni mean (BM) operators under DHPFS environment are proposed and some methods for multiple attribute decision-making (MADM) problems based on the BM operators with dual Pythagorean hesitant fuzzy numbers (DHPFNs) are investigated. Firstly, some new BM operators to aggregate dual Pythagorean hesitant fuzzy cognitive information are developed, which consider the interrelationship of DHPFNs, and can generate more accurate results than the existing dual Pythagorean hesitant fuzzy aggregation operators. Then, the developed aggregation operator is applied to MADM with DHPFNs and two MADM methods are designed, which can be applied to different decision-making situations. Based on the proposed operators and built models, two methods are developed to solve the MADM problems with DHPFNs and the validity and advantages of the proposed method are analyzed by comparison with some existing approaches. The methods proposed in this paper can effectively handle the MADM problems in which the attribute information is expressed by DHPFNs, the attributes' weights are completely known, and the attributes are interactive.

Keywords Multiple attribute decision-making (MADM) · Dual hesitant Pythagorean fuzzy sets · Dual hesitant Pythagorean fuzzy weighted Bonferroni mean (DHPFWBM) operator · Dual hesitant Pythagorean fuzzy weighted geometric Bonferroni mean (DHPFWGBM) operator · Supplier selection · Supply chain management

Introduction

Atanassov [1, 2] proposed the concept of intuitionistic fuzzy set (IFS), which is a generalization of fuzzy set [3]. The IFS has received more and more attention since its appearance [4–10]. Furthermore, Pythagorean fuzzy set (PFS) [11, 12] has appeared as an efficient tool for showing uncertainty of the multiple attribute decision-making (MADM) problems. The PFS is also characterized by the membership degree and the non-membership degree, whose sum of squares is less than or equal to 1. The PFS is more general than the IFS. Zhang and Xu [13] proposed the TOPSIS for solving the MADM problem within PFNs. Peng and Yang [14] mainly defined the division and subtraction operations of PFNs. Reformat and Yager [15] used

the PFNs to deal with the collaborative-based recommender system. Peng and Yang [14] developed a Pythagorean fuzzy superiority and inferiority ranking method to solve MAGDM with PFNs. Gou et al. [16] investigated the properties of continuous Pythagorean fuzzy information. Garg [17] proposed the new generalized Pythagorean fuzzy information aggregation by using Einstein operations. Wu and Wei [18] utilized Hamacher operations [19–23] to develop some Pythagorean fuzzy aggregation operators. Zeng et al. [24] developed a hybrid method for Pythagorean fuzzy MADM. Ren et al. [25] developed the Pythagorean fuzzy TODIM approach, which is useful to cope with the MADM problems that consider the DMs' psychological behaviors in uncertain circumstance. Wei and Lu [26] extended Maclaurin symmetric mean [27] to Pythagorean fuzzy environment to propose the Pythagorean fuzzy Maclaurin symmetric mean (PFMSM) operator and Pythagorean fuzzy weighted Maclaurin symmetric mean (PFWMSM) operator. Wei [28] utilized arithmetic and geometric operations [29–35] to develop some Pythagorean fuzzy interaction aggregation operators. Wei and Lu [36] utilized power aggregation operators

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[37, 38] to develop some Pythagorean fuzzy power aggregation operators. Wei and Wei [39] presented ten similarity measures between Pythagorean fuzzy sets (PFSs) based on the cosine function. Liang et al. [40] developed the Pythagorean fuzzy Bonferroni mean (PFBM) operator and weighted Pythagorean fuzzy Bonferroni mean (WPFBM) operator. Liang et al. [41] developed the Pythagorean fuzzy geometric Bonferroni mean (PFGBM) operator and weighted Pythagorean fuzzy geometric Bonferroni mean (WPFGBM) operators. Wei and Lu [42] proposed the concept and basic operations of the dual hesitant Pythagorean fuzzy sets (DHPFSs) based on the PFSs [11, 12] and dual hesitant fuzzy sets [43, 44] and developed some Hamacher aggregation operators for aggregating dual hesitant Pythagorean fuzzy information.

All the abovementioned operators and measures are proposed based on the assumption that input arguments which we want to aggregate are independent, and hence, in sometimes, they may be unable to justify the decision maker goals. On the other hand, in our real-life situation, it may be possible that there are interactions among the different attributes in a decision-making process. To address such type of issues, Bonferroni mean (BM) operator [45] and geometric Bonferroni mean (GBM) operator [46] have prominent characteristics to capture the interrelationship among the multi-input arguments. In the past few years, the BM and GBM have received more and more attentions and many important results both in theory and application are developed [47–57].

Therefore, by considering the advantages of the DHPFSs and the BM and GBM operators during the information fusion process, the present study enhanced the work in that direction. In it, DHPFSs have been used to handle the uncertainties in the data in the form of DHPFSs while BM and GBM operators are used to consider the interrelationships between the different attributes. As far as authors are aware that there is no research conducted under this direction, hence it is meaningful to pay some attention to it. So, to consider the advantages of interrelationships among any number of the attributes in DHPFSs environment, in this paper, we propose some Bonferroni mean aggregation operators for fusing the dual hesitant Pythagorean fuzzy information. Furthermore, some of their desirable properties have also been addressed. Finally, based on these operators, a decision-making approach has been presented under DHPFS environment and illustrated with a numerical example to validate the approach through some comparative studies with the existing approaches.

To do so, the rest of the paper is organized as follows. Some basic concepts on PFS and DHPFSs have been summarized in the next section. In “[Bonferroni Mean](#),” we propose the BM operators under DHPFS environment along with their certain properties. In “[GWBM Operator and GWGBM Operator](#),” we present the generalized weighted Bonferroni mean operator and generalized weighted geometric Bonferroni mean operator with DHPFNs. In “[DGWBM Operator and DGWGBM Operator](#),”

we present dual weighted Bonferroni mean operator and dual weighted geometric Bonferroni mean operator with DHPFNs. In “[Methods of MADM with DHPFNs](#),” we present some methods for MADM problems with DHPFNs. In “[Case Study and Comparative Analysis](#),” we present a numerical example for supplier selection with DHPFNs and give a comparative analysis with existing models. “[Conclusion](#)” concludes the paper with some remarks.

Preliminaries

Pythagorean Fuzzy Set

The basic definitions of PFSs [11, 12] are introduced in this section.

Definition 1 [11, 12] Let X be a fix set. A PFS is an object having the form

$$P = \{ \langle x, (\mu_P(x), \nu_P(x)) \rangle \mid x \in X \} \tag{1}$$

where the function $\mu_P : X \rightarrow [0, 1]$ defines the degree of membership and the function $\nu_P : X \rightarrow [0, 1]$ defines the degree of non-membership of the element $x \in X$ to P , respectively, and, for every $x \in X$, it holds that

$$(\mu_P(x))^2 + (\nu_P(x))^2 \leq 1 \tag{2}$$

Definition 2 [13] Let $\tilde{a}_1 = (\mu_1, \nu_1)$, $\tilde{a}_2 = (\mu_2, \nu_2)$, and $\tilde{a} = (\mu, \nu)$ be three Pythagorean fuzzy numbers, and some basic operations on them are defined as follows:

- (1) $\tilde{a}_1 \oplus \tilde{a}_2 = \left(\sqrt{(\mu_1)^2 + (\mu_2)^2 - (\mu_1)^2(\mu_2)^2}, \nu_1 \nu_2 \right)$;
- (2) $\tilde{a}_1 \otimes \tilde{a}_2 = \left(\mu_1 \mu_2, \sqrt{(\nu_1)^2 + (\nu_2)^2 - (\nu_1)^2(\nu_2)^2} \right)$;
- (3) $\pi \tilde{a} = \left(\sqrt{1 - (1 - \mu^2)^\pi}, \nu^\pi \right), \pi > 0$;
- (4) $(\tilde{a})^\pi = \left(\mu^\pi, \sqrt{1 - (1 - \nu^2)^\pi} \right), \pi > 0$;
- (5) $\tilde{a}^c = (\nu, \mu)$.

Dual Hesitant Pythagorean Fuzzy Set

Wei and Lu [42] proposed the concept and basic operations of the dual hesitant Pythagorean fuzzy sets (DHPFSs) based on the PFSs [11, 12] and dual hesitant fuzzy sets [43, 44].

Definition 3 [42] Let X be a fixed set, then a dual hesitant Pythagorean fuzzy set (DHPFS) on X is described as follows:

$$D = \{ \langle x, h_P(x), g_P(x) \rangle \mid x \in X \} \tag{3}$$

where $h_P(x)$ and $g_P(x)$ are two sets of some values in $[0, 1]$ that denote the possible membership degrees and non-membership degrees of the element $x \in X$ to the set P respectively, with the following conditions:

$$\gamma^2 + \eta^2 \leq 1 \tag{4}$$

where $\gamma \in h_P(x)$, $\eta \in g_P(x)$, for all $x \in X$. For convenience, the pair $d(x) = (h_P(x), g_P(x))$ is called a dual hesitant Pythagorean fuzzy number (DHPFN) denoted by $d = (h, g)$, with the following conditions: $\gamma \in h$, $\eta \in g$, $0 \leq \gamma, \eta \leq 1$, $0 \leq \gamma^2 + \eta^2 \leq 1$.

Furthermore, Wei and Lu [42] proposed the comparison laws between DHPFNs:

Definition 4 [42] Let $d = (h, g)$ be a DHPFN and $s(d) = \frac{1}{2} \left(1 + \frac{1}{\#h} \sum_{\gamma \in h} \gamma^2 - \frac{1}{\#g} \sum_{\eta \in g} \eta^2 \right)$ be the score function of $d = (h, g)$ and $p(d) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma^2 + \frac{1}{\#g} \sum_{\eta \in g} \eta^2$ be the accuracy function of $d = (h, g)$, where $\#h$ and $\#g$ are the numbers of the elements in h and g respectively; then, let $d_i = (h_i, g_i)$ ($i = 1, 2$) be any two DHPFNs; we have the following comparison laws:

- If $s(d_1) > s(d_2)$, then $d_1 \succ d_2$;
- If $s(d_1) = s(d_2)$, then

- (1) If $p(d_1) = p(d_2)$, then $d_1 = d_2$;
- (2) If $p(d_1) > p(d_2)$, then $d_1 \succ d_2$.

Then, Wei and Lu [42] defined some new operations on the DHPFNs d, d_1 , and d_2 :

$$(1) \quad d^\pi = \cup_{\gamma \in h, \eta \in g} \left\{ \{ \gamma^\pi \}, \left\{ \sqrt{1 - (1 - \eta^2)^\pi} \right\} \right\}, \pi > 0$$

$$(2) \quad \pi d = \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \sqrt{1 - (1 - \gamma^2)^\pi} \right\}, \{ \eta^\pi \} \right\}, \pi > 0$$

$$d_1 \oplus d_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ \sqrt{(\gamma_1)^2 + (\gamma_2)^2 - (\gamma_1)^2 (\gamma_2)^2} \right\}, \{ \eta_1 \eta_2 \} \right\}$$

$$d_1 \otimes d_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \{ \gamma_1 \gamma_2 \}, \left\{ \sqrt{(\eta_1)^2 + (\eta_2)^2 - (\eta_1)^2 (\eta_2)^2} \right\} \right\}.$$

Bonferroni Mean

Definition 5 [45] Let $p, q > 0$ and a_i ($i = 1, 2, \dots, n$) be a group of non-negative real numbers with the weights vector being $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, thereby satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. The weighted BM (WBM) operator is defined as follows:

$$WBM_{\omega}^{p,q}(a_1, a_2, \dots, a_n) = \left(\sum_{i,j=1}^n \omega_i \omega_j a_i^p a_j^q \right)^{1/(p+q)} \tag{5}$$

Definition 6 [46] Let $p, q > 0$ and a_i ($i = 1, 2, \dots, n$) be a group of non-negative real numbers with the weights vector being $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, thereby satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. If

$$WGBM_{\omega}^{p,q}(a_1, a_2, \dots, a_n) = \frac{1}{p+q} \prod_{i,j=1}^n (pa_i + qa_j)^{\omega_i \omega_j} \tag{6}$$

then $WGBM_{\omega}^{p,q}$ is called WGBM operator.

GWBM Operator and GWGBM Operator

Zhua et al. [46] proposed the generalized weighted BM (GWBM) operator and generalized weighted geometric BM (GWGBM) operator.

Definition 7 [46] Let $p, q, r > 0$ and a_i ($i = 1, 2, \dots, n$) be a group of non-negative real numbers with the weights vector being $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, $\omega_i \in [0, 1]$, and $\sum_{i=1}^n \omega_i = 1$. The generalized weighted BM (GWBM) operator is defined as follows:

$$GWBM_{\omega}^{p,q,r}(a_1, a_2, \dots, a_n) = \left(\sum_{i,j,k=1}^n \omega_i \omega_j \omega_k a_i^p a_j^q a_k^r \right)^{1/(p+q+r)} \tag{7}$$

Definition 8 [46] Let $p, q, r > 0$ and a_i ($i = 1, 2, \dots, n$) be a group of non-negative real numbers with the weights vector being $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, $\omega_i \in [0, 1]$, and $\sum_{i=1}^n \omega_i = 1$. If

$$GWGBM_{\omega}^{p,q,r}(a_1, a_2, \dots, a_n) = \frac{1}{p+q+r} \prod_{i,j,k=1}^n (pa_i + qa_j + ra_k)^{\omega_i \omega_j \omega_k} \tag{8}$$

then $GWGBM^{p,q,r}$ operator is called the generalized weighted geometric BM (GWGBM) operator.

DGWBm Operator and DGWGBM Operator

Zhang et al. [58] proposed the dual generalized weighted BM (DGWBm) operator and dual generalized weighted GBM (DGWGBM) operator.

Definition 9 [58] Let a_i ($i = 1, 2, \dots, n$) be a group of non-negative crisp numbers with the weight vector being $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, $\omega_i \in [0, 1]$ ($i = 1, 2, \dots, n$), and $\sum_{i=1}^n \omega_i = 1$. If

$$DGWBm_{\omega}^K(a_1, a_2, \dots, a_n) = \left(\sum_{i_1, i_2, \dots, i_n=1}^n \left(\prod_{j=1}^n \omega_{i_j} a_{i_j}^{k_j} \right) \right)^{1/\sum_{j=1}^n k_j} \tag{9}$$

then $K = (k_1, k_2, \dots, k_n)^T$ is parameter vector with $k_i \geq 0$ ($i = 1, 2, \dots, n$).

Definition 10 [58] Let $a_i(i = 1, 2, 3, \dots, n)$ be a group of non-negative numbers with the weight vector being $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T, \omega_i \in [0, 1] (i = 1, 2, 3, \dots, n)$, and $\sum_{i=1}^n \omega_i = 1$. If

$$DGWGBM_{\omega}^K(a_1, a_2, \dots, a_n) = \frac{1}{\sum_{j=1}^n k_j} \left(\prod_{i_1, i_2, \dots, i_n=1}^n \left(\sum_{j=1}^n (k_j p_{ij}) \right)^{\prod_{j=1}^n \omega_j} \right) \quad (10)$$

then $K = (k_1, k_2, \dots, k_n)^T$ is parameter vector with $k_i \geq 0 (i = 1, 2, 3, \dots, n)$.

DHPFWBM Operator and DHPFWGBM Operator

In this section, we propose the dual hesitant Pythagorean fuzzy weighted Bonferroni mean (DHPFWBM) operator and dual hesitant Pythagorean fuzzy weighted geometric Bonferroni mean (DHPFWGBM) operator based on the traditional Bonferroni mean operators.

DHPFWBM Operator

Definition 11 Let $t, r > 0, d_j = (h_j, g_j) (j = 1, 2, \dots, n)$ be a set of DHPFNs in which $h_P(x)$ and $g_P(x)$ are two sets of some values in $[0, 1]$ and weights vector being $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T, \omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$, then the DHPFWBM operator is defined as

$$DHPFWBM_{\omega}^{t,r}(d_1, d_2, \dots, d_n) = \left(\bigoplus_{i,j=1}^n \omega_i \omega_j (d_i^t \otimes d_j^r) \right)^{1/(t+r)} \quad (11)$$

Theorem 1 Let $t, r > 0$ and $d_j = (h_j, g_j) (j = 1, 2, \dots, n)$ be a collection of DHPFNs in which $\gamma_j \in h_j, \eta_j \in g_j$, then their aggregated value by using the DHPFWBM operator is also a DHPFN, and

$$\begin{aligned} DHPFWBM_{\omega}^{t,r}(d_1, d_2, \dots, d_n) &= \left(\bigoplus_{i,j=1}^n \omega_i \omega_j (d_i^t \otimes d_j^r) \right)^{1/(t+r)} \\ &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \left(\sqrt{1 - \prod_{i,j=1}^n (1 - \gamma_i^{2t} \gamma_j^{2r})^{\omega_i \omega_j}} \right)^{1/(t+r)} \right\}, \right. \\ &\quad \left. \left\{ \sqrt{1 - \left(1 - \prod_{i,j=1}^n (1 - (1 - \eta_i^2)^t (1 - \eta_j^2)^r \right)^{\omega_i \omega_j}} \right)^{1/(t+r)} \right\} \right\} \end{aligned} \quad (12)$$

Proof According to Definition 4, we can get

$$d_i^t = \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \{ \gamma_i^t \}, \left\{ \sqrt{1 - (1 - \eta_i^2)^t} \right\} \right\} \quad (13)$$

$$d_j^r = \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \{ \gamma_j^r \}, \left\{ \sqrt{1 - (1 - \eta_j^2)^r} \right\} \right\} \quad (14)$$

$$d_i^t \otimes d_j^r = \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \{ \gamma_i^t \gamma_j^r \}, \left\{ \sqrt{1 - (1 - \eta_i^2)^t (1 - \eta_j^2)^r} \right\} \right\} \quad (15)$$

Thereafter,

$$\begin{aligned} \omega_i \omega_j (d_i^t \otimes d_j^r) &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \sqrt{1 - (1 - \gamma_i^{2t} \gamma_j^{2r})^{\omega_i \omega_j}} \right\}, \left\{ \left(\sqrt{1 - (1 - \eta_i^2)^t (1 - \eta_j^2)^r} \right)^{\omega_i \omega_j} \right\} \right\} \end{aligned} \quad (16)$$

Furthermore,

$$\begin{aligned} \bigoplus_{i,j=1}^n \omega_i \omega_j (d_i^t \otimes d_j^r) &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \sqrt{1 - \prod_{i,j=1}^n (1 - \gamma_i^{2t} \gamma_j^{2r})^{\omega_i \omega_j}} \right\}, \left\{ \left(\sqrt{1 - (1 - \eta_i^2)^t (1 - \eta_j^2)^r} \right)^{\omega_i \omega_j} \right\} \right\} \end{aligned} \quad (17)$$

Therefore,

$$\begin{aligned} DHPFWBM_{\omega}^{t,r}(d_1, d_2, \dots, d_n) &= \left(\bigoplus_{i,j=1}^n \omega_i \omega_j (d_i^t \otimes d_j^r) \right)^{1/(t+r)} \\ &= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \left(\sqrt{1 - \prod_{i,j=1}^n (1 - \gamma_i^{2t} \gamma_j^{2r})^{\omega_i \omega_j}} \right)^{1/(t+r)} \right\}, \right. \\ &\quad \left. \left\{ \sqrt{1 - \left(1 - \prod_{i,j=1}^n (1 - (1 - \eta_i^2)^t (1 - \eta_j^2)^r \right)^{\omega_i \omega_j}} \right)^{1/(t+r)} \right\} \right\} \end{aligned} \quad (18)$$

Thereafter, we can get

$$0 \leq \left(\sqrt{1 - \prod_{i,j=1}^n (1 - \gamma_i^{2t} \gamma_j^{2r})^{\omega_i \omega_j}} \right)^{1/(t+r)} \leq 1 \quad (19)$$

$$0 \leq \sqrt{1 - \left(1 - \prod_{i,j=1}^n (1 - (1 - \eta_i^2)^t (1 - \eta_j^2)^r \right)^{\omega_i \omega_j}} \right)^{1/(t+r)} \leq 1 \quad (20)$$

And we know $\gamma^2 + \eta^2 \leq 1$, so

$$\left(\sqrt{1 - \prod_{i,j=1}^n (1 - \gamma_i^{2t} \gamma_j^{2r})^{\omega_i \omega_j}} \right)^{1/(t+r)} \leq \sqrt{\left(1 - \prod_{i,j=1}^n (1 - (1 - \eta_i^2)^t (1 - \eta_j^2)^r \right)^{\omega_i \omega_j}} \right)^{1/(t+r)} \quad (21)$$

Therefore,

$$\begin{aligned} &\left(\left(\sqrt{1 - \prod_{i,j=1}^n (1 - \gamma_i^{2t} \gamma_j^{2r})^{\omega_i \omega_j}} \right)^{1/(t+r)} \right)^2 + \\ &\left(\sqrt{1 - \left(1 - \prod_{i,j=1}^n (1 - (1 - \eta_i^2)^t (1 - \eta_j^2)^r \right)^{\omega_i \omega_j}} \right)^{1/(t+r)} \right)^2 \\ &\leq \left(1 - \prod_{i,j=1}^n (1 - (1 - \eta_i^2)^t (1 - \eta_j^2)^r \right)^{\omega_i \omega_j} \right)^{1/(t+r)} + 1 \\ &- \left(1 - \prod_{i,j=1}^n (1 - (1 - \eta_i^2)^t (1 - \eta_j^2)^r \right)^{\omega_i \omega_j} \right)^{1/(t+r)} = 1 \end{aligned} \quad (22)$$

So, we complete the proof.

Example 1 Let $a_1 = \{(0.3, 0.1), (0.4, 0.2)\}$, $a_2 = \{(0.4, 0.6), (0.5, 0.7)\}$, and $a_3 = \{(0.2, 0.5)\}$ be three DHPFNs with $\omega = (0.3, 0.5, 0.2)$, and $t = r = 2$, the aggregation result is as follows:

$$\begin{aligned} & \text{DHPFWBM}_{\omega}^{t,r}(a_1, a_2, a_3) \\ &= \text{DHPFWBM}_{\omega}^{t,r} \{ \{(0.3, 0.1), (0.4, 0.2)\}, \{(0.4, 0.6), (0.5, 0.7)\}, \{(0.2, 0.5)\} \} \\ &= \left\{ \left(\left(\left(\begin{array}{l} 1 - (1 - 0.3^{2 \times 2} \times 0.3^{2 \times 2})^{0.3 \times 0.3} \times \\ (1 - 0.3^{2 \times 2} \times 0.4^{2 \times 2})^{0.3 \times 0.5} \times \\ (1 - 0.3^{2 \times 2} \times 0.2^{2 \times 2})^{0.3 \times 0.2} \times \\ (1 - 0.4^{2 \times 2} \times 0.3^{2 \times 2})^{0.5 \times 0.3} \times \\ (1 - 0.4^{2 \times 2} \times 0.4^{2 \times 2})^{0.5 \times 0.5} \times \\ (1 - 0.4^{2 \times 2} \times 0.2^{2 \times 2})^{0.5 \times 0.2} \times \\ (1 - 0.2^{2 \times 2} \times 0.3^{2 \times 2})^{0.2 \times 0.3} \times \\ (1 - 0.2^{2 \times 2} \times 0.4^{2 \times 2})^{0.2 \times 0.5} \times \\ (1 - 0.2^{2 \times 2} \times 0.2^{2 \times 2})^{0.2 \times 0.2} \end{array} \right)^{1/(2+2)} \right. \right. \\ & \left. \left. \left(\left(\left(\begin{array}{l} 1 - (1 - 0.99^2 \times 0.99^2)^{0.3 \times 0.3} \times \\ (1 - 0.99^2 \times 0.64^2)^{0.3 \times 0.5} \times \\ (1 - 0.99^2 \times 0.75^2)^{0.3 \times 0.2} \times \\ (1 - 0.64^2 \times 0.99^2)^{0.5 \times 0.3} \times \\ (1 - 0.64^2 \times 0.64^2)^{0.5 \times 0.5} \times \\ (1 - 0.64^2 \times 0.75^2)^{0.5 \times 0.3} \times \\ (1 - 0.75^2 \times 0.99^2)^{0.2 \times 0.3} \times \\ (1 - 0.75^2 \times 0.64^2)^{0.2 \times 0.5} \times \\ (1 - 0.75^2 \times 0.75^2)^{0.2 \times 0.2} \end{array} \right)^{1/(2+2)} \right) \right. \right. \\ & \left. \left. \left. \right) \right\} \\ &= \{(0.3531, 0.4092), (0.4295, 0.4431), (0.3798, 0.4515), (0.4451, 0.4094)\} \end{aligned}$$

Then, we introduce three kinds of property of DHPFWBM.

Property 1 (Idempotency) Let $t, r > 0$ and $d_i = (h_i, g_i) (i = 1, 2, \dots, n)$ be two sets of DHPFNs. If $(i = 1, 2, \dots, n)$ are equal, that is, then

$$\text{DHPFWBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) = d \tag{23}$$

Proof

$$\begin{aligned} & \text{DHPFWBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) \\ &= \left(\bigoplus_{i,j=1}^n \omega_i \omega_j (d^t \otimes d^r) \right)^{1/(t+r)} \\ &= d \left(\bigoplus_{i,j=1}^n \omega_i \omega_j \right)^{1/(t+r)} \\ &= d \end{aligned} \tag{24}$$

Property 2 (Monotonicity) Let $d_j = (h_{d_j}, g_{d_j})$ and $b_j = (h_{b_j}, g_{b_j}) (j = 1, 2, 3, \dots, n)$ be two sets of DHPFNs.

If $\forall (\gamma_{d_j})^2 \leq \forall (\gamma_{b_j})^2, \gamma_{d_j} \in h_{d_j}, \gamma_{b_j} \in h_{b_j}$ and $\forall (\eta_{d_j})^2 \geq \forall (\eta_{b_j})^2, \eta_{d_j} \in g_{d_j}, \eta_{b_j} \in g_{b_j}$, then

$$\text{DHPFWBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) \leq \text{DHPFWBM}_{\omega}^{t,r}(b_1, b_2, \dots, b_n) \tag{25}$$

Proof We also can obtain,

$$\gamma_{d_i}^{2t} \gamma_{d_j}^{2r} \leq \gamma_{b_i}^{2t} \gamma_{b_j}^{2r} \tag{26}$$

$$\prod_{i,j=1}^n (1 - \gamma_{d_i}^{2t} \gamma_{d_j}^{2r}) \geq \prod_{i,j=1}^n (1 - \gamma_{b_i}^{2t} \gamma_{b_j}^{2r}) \tag{27}$$

$$1 - \prod_{i,j=1}^n (1 - \gamma_{d_i}^{2t} \gamma_{d_j}^{2r}) \leq 1 - \prod_{i,j=1}^n (1 - \gamma_{b_i}^{2t} \gamma_{b_j}^{2r}) \tag{28}$$

Therefore,

$$\left(\sqrt[1/(t+r)]{1 - \prod_{i,j=1}^n (1 - \gamma_{d_i}^{2t} \gamma_{d_j}^{2r})} \right) \leq \left(\sqrt[1/(t+r)]{1 - \prod_{i,j=1}^n (1 - \gamma_{b_i}^{2t} \gamma_{b_j}^{2r})} \right) \tag{29}$$

Thus,

$$\left(\left(\sqrt[1/(t+r)]{1 - \prod_{i,j=1}^n (1 - \gamma_{d_i}^{2t} \gamma_{d_j}^{2r})} \right)^2 \right) \leq \left(\left(\sqrt[1/(t+r)]{1 - \prod_{i,j=1}^n (1 - \gamma_{b_i}^{2t} \gamma_{b_j}^{2r})} \right)^2 \right) \tag{30}$$

which means $\gamma_d^2 \leq \gamma_b^2$, similarly, we can obtain $\eta_d^2 \geq \eta_b^2$.

If $\forall \gamma_{d_j}^2 < \forall \gamma_{b_j}^2$, and $\forall \eta_{d_j}^2 > \forall \eta_{b_j}^2$, then

$$\text{DHPFWBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n)$$

$$< \text{DHPFWBM}_{\omega}^{t,r}(b_1, b_2, \dots, b_n);$$

If $\forall \gamma_{d_j}^2 < \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 = \forall \eta_{b_j}^2$, then

$$\text{DHPFWBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) < \text{DHPFWBM}_{\omega}^{t,r}(b_1, b_2, \dots, b_n);$$

If $\forall \gamma_{d_j}^2 = \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 > \forall \eta_{b_j}^2$, then

$$\text{DHPFWBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) < \text{DHPFWBM}_{\omega}^{t,r}(b_1, b_2, \dots, b_n);$$

If $\forall \gamma_{d_j}^2 = \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 = \forall \eta_{b_j}^2$, then

$$\text{DHPFWBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) = \text{DHPFWBM}_{\omega}^{t,r}(b_1, b_2, \dots, b_n);$$

Therefore, the proof of Property 2 is completed.

Property 3 (Boundedness) Let $t, r > 0$ and $d_j = (h_{d_j}, g_{d_j}) (j = 1, 2, \dots, n)$ be a collection of DHPFNs.

If $d^+ = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\{\max_i(\gamma_i)\}, \{\min_i(\eta_i)\}\}$ and $d^- = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\{\min_i(\gamma_i)\}, \{\max_i(\eta_i)\}\}$, then

$$d^- \leq \text{DHPFWGBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) \leq d^+ \tag{31}$$

Proof From Property 1, we can obtain the following:

$$\begin{aligned} &\text{DHPFWGBM}_{\omega}^{t,r}(d^+, d^+, \dots, d^+) \\ &= d^+, \text{DHPFWGBM}_{\omega}^{t,r}(d^-, d^-, \dots, d^-) = d^- \end{aligned} \tag{32}$$

So, from Property 2, we can obtain the following:

$$\begin{aligned} d^- &= \text{DHPFWGBM}_{\omega}^{t,r}(d^-, d^-, \dots, d^-) \leq \\ &\text{DHPFWGBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) \leq \\ &\text{DHPFWGBM}_{\omega}^{t,r}(d^+, d^+, \dots, d^+) = d^+ \end{aligned} \tag{33}$$

DHPFWGBM Operator

In this section, we extend WGBM to DHPFN and introduced the dual hesitant Pythagorean fuzzy weighted geometric Bonferroni mean (DHPFWGBM) operator.

Definition 12 Let $t, r > 0, d_j = (h_j, g_j) (j = 1, 2, \dots, n)$ be a set of DHPFN in which $h_p(x)$ and $g_p(x)$ are two sets of some values in $[0, 1]$ and weights vector being $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T, \omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$, then the DHPFWGBM operator is defined as

$$\begin{aligned} &\text{DHPFWGBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) \\ &= \frac{1}{t+r} \otimes_{i,j=1}^n (td_i \oplus rd_j)^{\omega_i \omega_j} \end{aligned} \tag{34}$$

Theorem 2 Let $t, r > 0$ and $d_j = (h_j, g_j) (j = 1, 2, \dots, n)$ be a collection of DHPFNs in which $\gamma_j \in h_j, \eta_j \in g_j$, then their aggregated value by using the DHPFWGBM operator is also a DHPFN, and

$$\begin{aligned} &\text{DHPFWGBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) \\ &= \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - \left(1 - \prod_{i,j=1}^n \left(1 - (1 - \gamma_i^2)^t (1 - \gamma_j^2)^r \right)^{\omega_i \omega_j} \right)^{1/(t+r)}} \right) \right\}, \left\{ \left(\sqrt{1 - \prod_{i,j=1}^n \left(1 - \eta_i^{2t} \eta_j^{2r} \right)^{\omega_i \omega_j} \right)^{1/(t+r)}} \right\} \right\} \end{aligned} \tag{35}$$

Proof

$$td_i = \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \sqrt{1 - (1 - \gamma_i^2)^t} \right\}, \left\{ \eta_i^t \right\} \right\} \tag{36}$$

$$rd_j = \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \sqrt{1 - (1 - \gamma_j^2)^r} \right\}, \left\{ \eta_j^r \right\} \right\} \tag{37}$$

Thereafter,

$$td_i \oplus rd_j = \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \sqrt{1 - (1 - \gamma_i^2)^t (1 - \gamma_j^2)^r} \right\}, \left\{ \eta_i^t \eta_j^r \right\} \right\} \tag{38}$$

$$(td_i \oplus rd_j)^{\omega_i \omega_j} = \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - (1 - \gamma_i^2)^t (1 - \gamma_j^2)^r} \right)^{\omega_i \omega_j} \right\}, \left\{ \sqrt{1 - \eta_i^{2t} \eta_j^{2r}} \right\}^{\omega_i \omega_j} \right\} \tag{39}$$

Therefore,

$$\begin{aligned} \otimes_{i,j=1}^n (td_i \oplus rd_j)^{\omega_i \omega_j} &= \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - \prod_{i,j=1}^n \left(1 - (1 - \gamma_i^2)^t (1 - \gamma_j^2)^r \right)^{\omega_i \omega_j} \right) \right\}, \right. \\ &\left. \left\{ \sqrt{1 - \prod_{i,j=1}^n \left(1 - \eta_i^{2t} \eta_j^{2r} \right)^{\omega_i \omega_j} \right\} \right\} \end{aligned} \tag{40}$$

Thus,

$$\text{DHPFWGBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) = \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - \left(1 - \prod_{i,j=1}^n \left(1 - (1 - \gamma_i^2)^t (1 - \gamma_j^2)^r \right)^{\omega_i \omega_j} \right)^{1/(t+r)}} \right) \right\}, \left\{ \left(\sqrt{1 - \prod_{i,j=1}^n \left(1 - \eta_i^{2t} \eta_j^{2r} \right)^{\omega_i \omega_j} \right)^{1/(t+r)}} \right\} \right\} \tag{41}$$

Thereafter,

$$0 \leq \sqrt{1 - \left(1 - \prod_{i,j=1}^n \left(1 - (1 - \gamma_i^2)^t (1 - \gamma_j^2)^r\right)^{\omega_i \omega_j}\right)^{1/(t+r)}} \leq 1 \quad (42)$$

$$0 \leq \left(\sqrt{1 - \prod_{i,j=1}^n \left(1 - \eta_i^{2t} \eta_j^{2r}\right)^{\omega_i \omega_j}}\right)^{1/(t+r)} \leq 1 \quad (43)$$

Because of $\gamma^2 + \eta^2 \leq 1$,

$$\left(\sqrt{1 - \prod_{i,j=1}^n \left(1 - \eta_i^{2t} \eta_j^{2r}\right)^{\omega_i \omega_j}}\right)^{1/(t+r)} \leq \sqrt{\left(1 - \prod_{i,j=1}^n \left(1 - (1 - \gamma_i^2)^t (1 - \gamma_j^2)^r\right)^{\omega_i \omega_j}\right)^{1/(t+r)}} \quad (44)$$

Therefore,

$$\begin{aligned} & \left(\sqrt{1 - \left(1 - \prod_{i,j=1}^n \left(1 - (1 - \gamma_i^2)^t (1 - \gamma_j^2)^r\right)^{\omega_i \omega_j}\right)^{1/(t+r)}}\right)^2 + \\ & \left(\left(\sqrt{1 - \prod_{i,j=1}^n \left(1 - \eta_i^{2t} \eta_j^{2r}\right)^{\omega_i \omega_j}}\right)^{1/(t+r)}\right)^2 \leq \\ & 1 - \left(1 - \prod_{i,j=1}^n \left(1 - (1 - \gamma_i^2)^t (1 - \gamma_j^2)^r\right)^{\omega_i \omega_j}\right)^{1/(t+r)} \\ & + \left(1 - \prod_{i,j=1}^n \left(1 - (1 - \gamma_i^2)^t (1 - \gamma_j^2)^r\right)^{\omega_i \omega_j}\right)^{1/(t+r)} = 1 \end{aligned} \quad (45)$$

thereby completing the proof.

Example 2 Let $a_1 = \{(0.3, 0.1), (0.4, 0.2)\}, a_2 = \{(0.4, 0.6), (0.5, 0.7)\}$, and $a_3 = \{(0.2, 0.5)\}$ be three DHPFNs with $\omega = (0.3, 0.5, 0.2)$, and $t = r = 2$, the aggregation result as follows:

$$\begin{aligned} & \text{DHPFWGBM}_{\omega}^{t,r}(a_1, a_2, a_3) \\ & = \text{DHPFWGBM}_{\omega}^{t,r} \{ \{(0.3, 0.1), (0.4, 0.2)\}, \{(0.4, 0.6), (0.5, 0.7)\}, \{(0.2, 0.5)\} \} \\ & = \left\{ \left(\sqrt{1 - \left(\prod_{i,j=1}^n \left(1 - (1 - 0.91^2 \times 0.91^2)^{0.3 \times 0.3} \times \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left(1 - 0.91^2 \times 0.84^2 \right)^{0.3 \times 0.5} \times \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left(1 - 0.91^2 \times 0.96^2 \right)^{0.3 \times 0.2} \times \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left(1 - 0.84^2 \times 0.91^2 \right)^{0.5 \times 0.3} \times \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left(1 - 0.84^2 \times 0.84^2 \right)^{0.5 \times 0.5} \times \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left(1 - 0.84^2 \times 0.96^2 \right)^{0.5 \times 0.2} \times \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left(1 - 0.96^2 \times 0.91^2 \right)^{0.2 \times 0.3} \times \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left(1 - 0.96^2 \times 0.84^2 \right)^{0.2 \times 0.5} \times \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left(1 - 0.96^2 \times 0.96^2 \right)^{0.2 \times 0.2} \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. \left. \right. \right. \right. \right. \right. \right. \left. \right\}, \dots \left. \right\} \\ & = \{(0.3295, 0.5276), (0.3765, 0.6047), (0.3601, 0.5284), (0.4087, 0.6052)\} \end{aligned}$$

Similar to DHPFWBM, the DHPFWGBM has the same properties. The proofs of these properties are similar to that of the properties of DHPFWGBM. Accordingly, the proofs are omitted to save space.

Property 4 (Idempotency) Let $t, r > 0$ and $d_i = (h_i, g_i) (i = 1, 2, \dots, n)$ be two sets of DHPFNs, if $d_i (i = 1, 2, \dots, n)$ are equal, that is $d_i = d = (h, g)$, then

$$\text{DHPFWGBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) = d \quad (46)$$

Property 5 (Monotonicity) Let $d_j = (h_{d_j}, g_{d_j})$ and $b_j = (h_{b_j}, g_{b_j}) (j = 1, 2, \dots, n)$ be two sets of DHPFNs,

$$\begin{aligned} & \text{If } \forall (\gamma_{d_j})^2 \leq (\gamma_{b_j})^2, \gamma_{d_j} \in h_{d_j}, \gamma_{b_j} \in h_{b_j} \quad \text{and} \\ & \forall (\eta_{d_j})^2 \geq (\eta_{b_j})^2, \eta_{d_j} \in g_{d_j}, \eta_{b_j} \in g_{b_j} \quad \text{then} \end{aligned}$$

$$\text{DHPFWGBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) \leq \text{DHPFWGBM}_{\omega}^{t,r}(b_1, b_2, \dots, b_n) \quad (47)$$

Property 6 (Boundedness) Let $t, r > 0$ and $d_j = (h_{d_j}, g_{d_j}) (j = 1, 2, 3, \dots, n)$ be a collection of DHPFNs. If $d^+ = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{ \{\max_i(\gamma_i)\}, \{\min_i(\eta_i)\} \}$ and $d^- = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{ \{\min_i(\gamma_i)\}, \{\max_i(\eta_i)\} \}$, then

$$d \leq \text{DHPFWGBM}_{\omega}^{t,r}(d_1, d_2, \dots, d_n) \leq d^+ \quad (48)$$

DHPFGWBM Operator and DHPFGWGBM Operator

In this section, we combine DHPFNs with generalized Bonferroni mean operators to propose the dual hesitant Pythagorean fuzzy generalized weighted Bonferroni mean (DHPFGWBM) operator and dual hesitant Pythagorean fuzzy generalized weighted geometric Bonferroni mean (DHPFGWGBM) operator.

DHPFGWBM Operator

Definition 13 Lett, $r > 0, d_j = (h_j, g_j) (j = 1, 2, \dots, n)$ be a set of DHPFN in which $h_P(x)$ and $g_P(x)$ are two sets of some values in $[0, 1]$ and weights vector being $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T, \omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$. If

$$DHPFGWBM_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) = \left(\bigoplus_{i,j,k=1}^n \omega_i \omega_j \omega_k d_i^s \otimes d_j^t \otimes d_k^r \right)^{1/(\alpha+\beta+\gamma)} \quad (49)$$

Theorem 3 Lets, $t, r > 0$ and $d_j = (h_j, g_j) (j = 1, 2, \dots, n)$ be a collection of DHPFNs. The aggregated value by DHPFGWBM is also a DHPFN and

$$\begin{aligned} & DHPFGWBM_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \\ &= \left(\bigoplus_{i,j,k=1}^n \omega_i \omega_j \omega_k d_i^s \otimes d_j^t \otimes d_k^r \right)^{1/(\alpha+\beta+\gamma)} \\ &= \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \sqrt{1 - \prod_{i,j,k=1}^n (1 - \gamma_i^{2s} \gamma_j^{2t} \gamma_k^{2r})^{\omega_i \omega_j \omega_k}} \right\}^{1/(s+t+r)} \right\}, \\ & \left\{ \sqrt{1 - \left(1 - \prod_{i,j,k=1}^n (1 - (1 - \eta_i^2)^s (1 - \eta_j^2)^t (1 - \eta_k^2)^r \right)^{\omega_i \omega_j \omega_k}} \right\}^{1/(s+t+r)} \end{aligned} \quad (50)$$

Proof According to Definition 4, we can obtain

$$d_i^s = \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \{ \gamma_i^s \}, \left\{ \sqrt{1 - (1 - \eta_i^2)^s} \right\} \right\} \quad (51)$$

$$d_j^t = \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \{ \gamma_j^t \}, \left\{ \sqrt{1 - (1 - \eta_j^2)^t} \right\} \right\} \quad (52)$$

$$d_k^r = \cup_{\gamma_k \in h_k, \eta_k \in g_k} \left\{ \{ \gamma_k^r \}, \left\{ \sqrt{1 - (1 - \eta_k^2)^r} \right\} \right\} \quad (53)$$

Thus,

$$d_i^s \otimes d_j^t \otimes d_k^r = \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \{ \gamma_i^s \gamma_j^t \gamma_k^r \}, \left\{ \sqrt{1 - (1 - \eta_i^2)^s (1 - \eta_j^2)^t (1 - \eta_k^2)^r} \right\} \right\} \quad (54)$$

Thereafter,

$$\begin{aligned} & \omega_i \omega_j \omega_k (d_i^s \otimes d_j^t \otimes d_k^r) \\ &= \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \sqrt{1 - (1 - \gamma_i^{2s} \gamma_j^{2t} \gamma_k^{2r})^{\omega_i \omega_j \omega_k}} \right\}, \right. \\ & \left. \left\{ \left(\sqrt{1 - (1 - \eta_i^2)^s (1 - \eta_j^2)^t (1 - \eta_k^2)^r} \right)^{\omega_i \omega_j \omega_k} \right\} \right\} \end{aligned} \quad (55)$$

Furthermore,

$$\begin{aligned} & \bigoplus_{i,j,k=1}^n \omega_i \omega_j \omega_k (d_i^s \otimes d_j^t \otimes d_k^r) \\ &= \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \sqrt{1 - \prod_{i,j,k=1}^n (1 - \gamma_i^{2s} \gamma_j^{2t} \gamma_k^{2r})^{\omega_i \omega_j \omega_k}} \right\}, \right. \\ & \left. \left\{ \left(\sqrt{1 - \prod_{i,j,k=1}^n (1 - (1 - \eta_i^2)^s (1 - \eta_j^2)^t (1 - \eta_k^2)^r} \right)^{\omega_i \omega_j \omega_k} \right\} \right\} \end{aligned} \quad (56)$$

Therefore,

$$\begin{aligned} & DHPFGWBM_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \\ &= \left(\bigoplus_{i,j,k=1}^n \omega_i \omega_j \omega_k d_i^s \otimes d_j^t \otimes d_k^r \right)^{1/(\alpha+\beta+\gamma)} \\ &= \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \left(\sqrt{1 - \prod_{i,j,k=1}^n (1 - \gamma_i^{2s} \gamma_j^{2t} \gamma_k^{2r})^{\omega_i \omega_j \omega_k}} \right)^{1/(s+t+r)} \right\}, \right. \\ & \left. \left\{ \sqrt{1 - \left(1 - \prod_{i,j,k=1}^n (1 - (1 - \eta_i^2)^s (1 - \eta_j^2)^t (1 - \eta_k^2)^r \right)^{\omega_i \omega_j \omega_k}} \right\}^{1/(s+t+r)} \right\} \end{aligned} \quad (57)$$

Hence, (50) is maintained. Thereafter,

$$0 \leq \left(\sqrt{1 - \prod_{i,j,k=1}^n (1 - \gamma_i^{2s} \gamma_j^{2t} \gamma_k^{2r})^{\omega_i \omega_j \omega_k}} \right)^{1/(s+t+r)} \leq 1 \quad (58)$$

$$0 \leq \sqrt{1 - \left(1 - \prod_{i,j,k=1}^n (1 - (1 - \eta_i^2)^s (1 - \eta_j^2)^t (1 - \eta_k^2)^r \right)^{\omega_i \omega_j \omega_k}} \right)^{1/(s+t+r)} \leq 1 \quad (59)$$

Because $\gamma^2 + \eta^2 \leq 1$,

$$\begin{aligned} & \left(\sqrt{1 - \prod_{i,j,k=1}^n (1 - \gamma_i^{2s} \gamma_j^{2t} \gamma_k^{2r})^{\omega_i \omega_j \omega_k}} \right)^{1/(s+t+r)} \leq \\ & \sqrt{\left(1 - \prod_{i,j,k=1}^n (1 - (1 - \eta_i^2)^s (1 - \eta_j^2)^t (1 - \eta_k^2)^r \right)^{\omega_i \omega_j \omega_k}} \right)^{1/(s+t+r)} \end{aligned} \quad (60)$$

Therefore,

$$\begin{aligned} & \left(\left(\sqrt{1 - \prod_{i,j,k=1}^n (1 - \gamma_i^{2s} \gamma_j^{2t} \gamma_k^{2r})^{\omega_i \omega_j \omega_k}} \right)^{1/(s+t+r)} \right)^2 + \\ & \left(\sqrt{1 - \left(1 - \prod_{i,j,k=1}^n (1 - (1 - \eta_i^2)^s (1 - \eta_j^2)^t (1 - \eta_k^2)^r \right)^{\omega_i \omega_j \omega_k}} \right)^{1/(s+t+r)} \right)^2 \\ & \leq \left(1 - \prod_{i,j,k=1}^n (1 - (1 - \eta_i^2)^s (1 - \eta_j^2)^t (1 - \eta_k^2)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)} + 1 \\ & - \left(1 - \prod_{i,j,k=1}^n (1 - (1 - \eta_i^2)^s (1 - \eta_j^2)^t (1 - \eta_k^2)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)} = 1 \end{aligned} \quad (61)$$

thereby completing the proof.

Furthermore, DHPFGWBM has the following properties.

Property 7 (Idempotency) If $d_i (i = 1, 2, \dots, n)$ are equal, that is $d_i = d = (h, g)$, then

$$\text{DHPFGWBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) = d \tag{62}$$

Proof

$$\begin{aligned} &\text{DHPFGWBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \\ &= \left(\bigoplus_{i,j,k=1}^n \omega_i \omega_j \omega_k \left(d_i^s \otimes d_j^t \otimes d_k^r \right) \right)^{1/(s+t+r)} \\ &= d \left(\bigoplus_{i,j,k=1}^n \omega_i \omega_j \omega_k \right)^{1/(s+t+r)} \\ &= d \end{aligned} \tag{63}$$

Property 8 (Monotonicity) Let $d_j = (h_{d_j}, g_{d_j})$ and $b_j = (h_{b_j}, g_{b_j})$ ($j = 1, 2, \dots, n$) be two sets of DHPFNs.

If $\forall (\gamma_{d_j})^2 \leq \forall (\gamma_{b_j})^2, \gamma_{d_j} \in h_{d_j}, \gamma_{b_j} \in h_{b_j}$ and $\forall (\eta_{d_j})^2 \geq \forall (\eta_{b_j})^2, \eta_{d_j} \in g_{d_j}, \eta_{b_j} \in g_{b_j}$, then

$$\text{DHPFGWBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \leq \text{DHPFGWBM}_{\omega}^{s,t,r}(b_1, b_2, \dots, b_n) \tag{64}$$

Proof We can obtain

$$\gamma_{d_i}^{2s} \gamma_{d_j}^{2t} \gamma_{d_k}^{2r} \leq \gamma_{b_i}^{2s} \gamma_{b_j}^{2t} \gamma_{b_k}^{2r} \tag{65}$$

$$\prod_{i,j,k=1}^n \left(1 - \gamma_{d_i}^{2s} \gamma_{d_j}^{2t} \gamma_{d_k}^{2r} \right) \geq \prod_{i,j,k=1}^n \left(1 - \gamma_{b_i}^{2s} \gamma_{b_j}^{2t} \gamma_{b_k}^{2r} \right) \tag{66}$$

$$1 - \prod_{i,j,k=1}^n \left(1 - \gamma_{d_i}^{2s} \gamma_{d_j}^{2t} \gamma_{d_k}^{2r} \right) \leq 1 - \prod_{i,j,k=1}^n \left(1 - \gamma_{b_i}^{2s} \gamma_{b_j}^{2t} \gamma_{b_k}^{2r} \right) \tag{67}$$

Therefore,

$$\left(\sqrt[1/(s+t+r)]{1 - \prod_{i,j,k=1}^n \left(1 - \gamma_{d_i}^{2s} \gamma_{d_j}^{2t} \gamma_{d_k}^{2r} \right)} \right) \leq \left(\sqrt[1/(s+t+r)]{1 - \prod_{i,j,k=1}^n \left(1 - \gamma_{b_i}^{2s} \gamma_{b_j}^{2t} \gamma_{b_k}^{2r} \right)} \right) \tag{68}$$

Thus,

$$\left(\left(\sqrt[1/(s+t+r)]{1 - \prod_{i,j,k=1}^n \left(1 - \gamma_{d_i}^{2s} \gamma_{d_j}^{2t} \gamma_{d_k}^{2r} \right)} \right)^2 \right) \leq \left(\left(\sqrt[1/(s+t+r)]{1 - \prod_{i,j,k=1}^n \left(1 - \gamma_{b_i}^{2s} \gamma_{b_j}^{2t} \gamma_{b_k}^{2r} \right)} \right)^2 \right) \tag{69}$$

which means $\forall \gamma_d^2 \leq \forall \gamma_b^2$, similarly, we can obtain $\forall \eta_d^2 \geq \forall \eta_b^2$.

If $\forall \gamma_{d_j}^2 < \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 > \forall \eta_{b_j}^2$, then

$$\begin{aligned} &\text{DHPFGWBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \\ &< \text{DHPFGWBM}_{\omega}^{s,t,r}(b_1, b_2, \dots, b_n); \end{aligned}$$

If $\forall \gamma_{d_j}^2 < \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 = \forall \eta_{b_j}^2$, then

$$\begin{aligned} &\text{DHPFGWBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \\ &< \text{DHPFGWBM}_{\omega}^{s,t,r}(b_1, b_2, \dots, b_n); \end{aligned}$$

If $\forall \gamma_{d_j}^2 = \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 > \forall \eta_{b_j}^2$, then

$$\begin{aligned} &\text{DHPFGWBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \\ &< \text{DHPFGWBM}_{\omega}^{s,t,r}(b_1, b_2, \dots, b_n); \end{aligned}$$

If $\forall \gamma_{d_j}^2 = \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 = \forall \eta_{b_j}^2$, then

$$\begin{aligned} &\text{DHPFGWBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \\ &= \text{DHPFGWBM}_{\omega}^{s,t,r}(b_1, b_2, \dots, b_n); \end{aligned}$$

Therefore, the proof of Property 8 is completed.

Property 9 (Boundedness) Let $t, r > 0$ and $d_j = (h_{d_j}, g_{d_j})$ ($j = 1, 2, \dots, n$) be a collection of DHPFNs.

If $d^+ = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{ \max_i(\gamma_i), \min_i(\eta_i) \}$ and $d^- = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{ \min_i(\gamma_i), \max_i(\eta_i) \}$, then

$$d^- \leq \text{DHPFGWBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \leq d^+ \tag{70}$$

Proof From Property 7, we can obtain

$$\begin{aligned} &\text{DHPFGWBM}_{\omega}^{s,t,r}(d^+, d^+, \dots, d^+) \\ &= d^+ \text{DHPFGWBM}_{\omega}^{s,t,r}(d^-, d^-, \dots, d^-) = d^- \end{aligned} \tag{71}$$

From Property 8, we can obtain

$$\begin{aligned} &d^- = \text{DHPFGWBM}_{\omega}^{s,t,r}(d^-, d^-, \dots, d^-) \leq \\ &\text{DHPFGWBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \leq \\ &\text{DHPFGWBM}_{\omega}^{s,t,r}(d^+, d^+, \dots, d^+) = d^+ \end{aligned} \tag{72}$$

Therefore,

$$d^- \leq \text{DHPFGWBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \leq d^+ \tag{73}$$

DHPFGWGBM Operator

Thereafter, we extend GWGBM operator [46] to DHPFN and propose the dual hesitant Pythagorean fuzzy generalized weighted geometric Bonferroni mean (DHPFGWGBM) operator.

Definition 14 Lets, $t, r > 0, d_j = (h_j, g_j) (j = 1, 2, \dots, n)$ be a set of DHPFN in which $h_p(x)$ and $g_p(x)$ are two sets of some values in $[0, 1]$ and weights vector being $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T, \omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$. If

$$\begin{aligned} & \text{DHPFGWGBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \\ &= \frac{1}{s+t+r} \bigotimes_{i,j,k=1}^n (sd_i \oplus td_j \oplus rd_k)^{\omega_i \omega_j \omega_k} \end{aligned} \tag{74}$$

Theorem 4 Lets, $t, r > 0$ and $d_j = (h_j, g_j) (j = 1, 2, \dots, n)$ be a collection of DHPFNs. The aggregated value by DHPFGWGBM is also a DHPFN and

$$\begin{aligned} & \text{DHPFGWGBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \\ &= \frac{1}{s+t+r} \bigotimes_{i,j,k=1}^n (sd_i \oplus td_j \oplus rd_k)^{\omega_i \omega_j \omega_k} \\ &= \cup_{\gamma \in h, \eta \in g} \left\{ \left(\sqrt{1 - \left(1 - \prod_{i,j,k=1}^n \left(1 - (1 - \gamma_i^2)^s (1 - \gamma_j^2)^t (1 - \gamma_k^2)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)}} \right. \right. \\ & \quad \left. \left. \left\{ \left(\sqrt{1 - \prod_{i,j,k=1}^n \left(1 - \eta_i^{2\alpha} \eta_j^{2\beta} \eta_k^{2\gamma} \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)}} \right\} \right\} \end{aligned} \tag{75}$$

Proof Through Definition 4, we can obtain

$$sd_i = \cup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ \sqrt{1 - (1 - \gamma_i^2)^s} \right\}, \left\{ \eta_i^s \right\} \right\} \tag{76}$$

$$td_j = \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \sqrt{1 - (1 - \gamma_j^2)^t} \right\}, \left\{ \eta_j^t \right\} \right\} \tag{77}$$

$$rd_k = \cup_{\gamma_k \in h_k, \eta_k \in g_k} \left\{ \left\{ \sqrt{1 - (1 - \gamma_k^2)^r} \right\}, \left\{ \eta_k^r \right\} \right\} \tag{78}$$

Thereafter,

$$sd_i \oplus td_j \oplus rd_k = \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \sqrt{1 - (1 - \gamma_i^2)^s (1 - \gamma_j^2)^t (1 - \gamma_k^2)^r} \right\}, \left\{ \eta_i^s \eta_j^t \eta_k^r \right\} \right\} \tag{79}$$

Thereafter,

$$\begin{aligned} & (sd_i \oplus td_j \oplus rd_k)^{\omega_i \omega_j \omega_k} \\ &= \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - (1 - \gamma_i^2)^s (1 - \gamma_j^2)^t (1 - \gamma_k^2)^r} \right)^{\omega_i \omega_j \omega_k} \right\}, \right. \\ & \quad \left. \left\{ \sqrt{1 - \left(1 - \eta_i^{2s} \eta_j^{2t} \eta_k^{2r} \right)^{\omega_i \omega_j \omega_k}} \right\} \right\} \end{aligned} \tag{80}$$

Therefore,

$$\begin{aligned} & \bigotimes_{i,j,k=1}^n (sd_i \oplus td_j \oplus rd_k)^{\omega_i \omega_j \omega_k} \\ &= \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - \prod_{i,j,k=1}^n \left(1 - (1 - \gamma_i^2)^s (1 - \gamma_j^2)^t (1 - \gamma_k^2)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)} \right\}, \right. \\ & \quad \left. \left\{ \sqrt{1 - \prod_{i,j,k=1}^n \left(1 - \eta_i^{2s} \eta_j^{2t} \eta_k^{2r} \right)^{\omega_i \omega_j \omega_k}} \right\} \right\} \end{aligned} \tag{81}$$

Thus,

$$\begin{aligned} & \text{DHPFGWGBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \\ &= \frac{1}{s+t+r} \bigotimes_{i,j,k=1}^n (sd_i \oplus td_j \oplus rd_k)^{\omega_i \omega_j \omega_k} \\ &= \cup_{\gamma \in h, \eta \in g} \left\{ \left(\sqrt{1 - \left(1 - \prod_{i,j,k=1}^n \left(1 - (1 - \gamma_i^2)^s (1 - \gamma_j^2)^t (1 - \gamma_k^2)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)}} \right. \right. \\ & \quad \left. \left. \left\{ \left(\sqrt{1 - \prod_{i,j,k=1}^n \left(1 - \eta_i^{2\alpha} \eta_j^{2\beta} \eta_k^{2\gamma} \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)}} \right\} \right\} \end{aligned} \tag{82}$$

Hence, (75) is maintained. Thereafter,

$$0 \leq \sqrt{1 - \left(1 - \prod_{i,j,k=1}^n \left(1 - (1 - \gamma_i^2)^s (1 - \gamma_j^2)^t (1 - \gamma_k^2)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)}} \leq 1 \tag{83}$$

$$0 \leq \left(\sqrt{1 - \prod_{i,j,k=1}^n \left(1 - \eta_i^{2\alpha} \eta_j^{2\beta} \eta_k^{2\gamma} \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)} \leq 1 \tag{84}$$

Because $\gamma^2 + \eta^2 \leq 1$,

$$\begin{aligned} & \left(\sqrt{1 - \prod_{i,j,k=1}^n \left(1 - \eta_i^{2s} \eta_j^{2t} \eta_k^{2r} \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)} \leq \\ & \sqrt{\left(1 - \prod_{i,j,k=1}^n \left(1 - (1 - \gamma_i^2)^s (1 - \gamma_j^2)^t (1 - \gamma_k^2)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)}} \end{aligned} \tag{85}$$

Therefore,

$$\begin{aligned} & \left(\sqrt{1 - \left(1 - \prod_{i,j,k=1}^n \left(1 - (1 - \gamma_i^2)^s (1 - \gamma_j^2)^t (1 - \gamma_k^2)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)}} \right)^2 + \\ & \left(\left(\sqrt{1 - \prod_{i,j,k=1}^n \left(1 - \eta_i^{2s} \eta_j^{2t} \eta_k^{2r} \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)} \right)^2 \leq \\ & 1 - \left(1 - \prod_{i,j,k=1}^n \left(1 - (1 - \gamma_i^2)^s (1 - \gamma_j^2)^t (1 - \gamma_k^2)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)} \\ & + \left(1 - \prod_{i,j,k=1}^n \left(1 - (1 - \gamma_i^2)^s (1 - \gamma_j^2)^t (1 - \gamma_k^2)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(s+t+r)} = 1 \end{aligned} \tag{86}$$

thereby completing the proof.

Similar to DHPFGWBM operator, the DHPFGWGBM operator has the same properties. The proofs of these properties are similar to that of the properties of DHPFGWGBM. Accordingly, the proofs are omitted to save space.

Property 10 (Idempotency) If $d_i (i = 1, 2, \dots, n)$ are equal, that is $d_i = d = (h, g)$, then

$$\text{DHPFGWGBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) = d \tag{87}$$

Property 11 (Monotonicity) Let $d_j = (h_{d_j}, g_{d_j})$ and $b_j = (h_{b_j}, g_{b_j})$ ($j = 1, 2, 3, \dots, n$) be two sets of DHPFNs.

$$\text{If } \forall (\gamma_{d_j})^2 \leq \forall (\gamma_{b_j})^2, \gamma_{d_j} \in h_{d_j}, \gamma_{b_j} \in h_{b_j} \quad \text{and} \\ \forall (\eta_{d_j})^2 \geq \forall (\eta_{b_j})^2, \eta_{d_j} \in g_{d_j}, \eta_{b_j} \in g_{b_j}, \text{ then}$$

$$\text{DHPFGWGBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \leq \text{DHPFGWGBM}_{\omega}^{s,t,r}(b_1, b_2, \dots, b_n) \quad (88)$$

Property 12 (Boundedness) Let $t, r > 0$ and $d_j = (h_{d_j}, g_{d_j})$ ($j = 1, 2, 3, \dots, n$) be a collection of DHPFNs. If $d^+ = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\max_i(\gamma_i), \{\min_i(\eta_i)\}\}$ and $d^- = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\min_i(\gamma_i), \{\max_i(\eta_i)\}\}$, then

$$d^- \leq \text{DHPFGWGBM}_{\omega}^{s,t,r}(d_1, d_2, \dots, d_n) \leq d^+ \quad (89)$$

DHPFDWBM Operator and DHPFDWGBM Operator

In this section, we go on with deriving the dual hesitant Pythagorean fuzzy dual weighted Bonferroni mean (DHPFDWBM) operator and dual hesitant Pythagorean fuzzy dual weighted geometric Bonferroni mean (DHPFDWGBM) operator.

DHPFDWBM Operator

Definition 15 Let $l_j > 0$ and $d_j = (h_j, g_j)$ ($j = 1, 2, \dots, n$) be a set of DHPFNs in which $h_P(x)$ and $g_P(x)$ are two sets of some values in $[0, 1]$ and weights vector being $W = (\omega_1, \omega_2, \dots, \omega_n)^T, \omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$. If

$$\text{DHPFDWBM}_{\omega}^l(d_1, d_2, \dots, d_n) = \left(\bigoplus_{i_1, i_2, \dots, i_n=1}^n \left(\bigotimes_{j=1}^n \omega_{i_j} d_{i_j}^{l_j} \right) \right)^{1/\sum_{i=1}^n l_i} \quad (90)$$

Theorem 5 Let $l_j > 0$ and $d_j = (h_j, g_j)$ ($j = 1, 2, \dots, n$) be a collection of DHPFNs. The aggregated result of DHPFDWBM operator is a DHPFN.

$$\text{DHPFDWBM}_{\omega}^l(d_1, d_2, \dots, d_n) = \left(\bigoplus_{i_1, i_2, \dots, i_n=1}^n \left(\bigotimes_{j=1}^n \omega_{i_j} d_{i_j}^{l_j} \right) \right)^{1/\sum_{i=1}^n l_i} \\ = \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left(\sqrt{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - (1 - \gamma_{i_j}^{2l_j})^{\omega_{i_j}} \right) \right)} \right)^{1/\sum_{i=1}^n l_i} \right\}, \\ \left\{ \sqrt{1 - \left(1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - (1 - \eta_{i_j}^{2l_j})^{\omega_{i_j}} \right) \right) \right)} \right)^{1/\sum_{i=1}^n l_i} \right\} \quad (91)$$

Proof

$$d_{i_j}^{l_j} = \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left(\gamma_{i_j} \right)^{l_j}, \left\{ \sqrt{1 - (1 - \eta_{i_j}^2)^{l_j}} \right\} \right\} \quad (92)$$

$$\omega_{i_j} d_{i_j}^{l_j} = \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \sqrt{1 - (1 - \gamma_{i_j}^{2l_j})^{\omega_{i_j}}} \right\}, \left\{ \left(\sqrt{1 - (1 - \eta_{i_j}^2)^{l_j}} \right)^{\omega_{i_j}} \right\} \right\} \quad (93)$$

Thus,

$$\bigotimes_{j=1}^n \omega_{i_j} d_{i_j}^{l_j} = \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left(\left\{ \left\{ \prod_{j=1}^n \sqrt{1 - (1 - \gamma_{i_j}^{2l_j})^{\omega_{i_j}}} \right\}, \left(\sqrt{1 - \prod_{j=1}^n \left(1 - (1 - \eta_{i_j}^2)^{l_j} \right)^{\omega_{i_j}}} \right) \right\} \right) \quad (94)$$

Thereafter,

$$\bigoplus_{i_1, i_2, \dots, i_n=1}^n \left(\bigotimes_{j=1}^n \omega_{i_j} d_{i_j}^{l_j} \right) = \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \sqrt{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - (1 - \gamma_{i_j}^{2l_j})^{\omega_{i_j}} \right) \right)} \right\}, \left\{ \prod_{i_1, i_2, \dots, i_n=1}^n \sqrt{1 - \prod_{j=1}^n \left(1 - (1 - \eta_{i_j}^2)^{l_j} \right)^{\omega_{i_j}}} \right\} \right\} \quad (95)$$

Therefore,

$$\text{DHPFDWBM}_{\omega}^l(d_1, d_2, \dots, d_n) = \left(\bigoplus_{i_1, i_2, \dots, i_n=1}^n \left(\bigotimes_{j=1}^n \omega_{i_j} d_{i_j}^{l_j} \right) \right)^{1/\sum_{i=1}^n l_i} \\ = \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left(\sqrt{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - (1 - \gamma_{i_j}^{2l_j})^{\omega_{i_j}} \right) \right)} \right)^{1/\sum_{i=1}^n l_i} \right\}, \\ \left\{ \sqrt{1 - \left(1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - (1 - \eta_{i_j}^2)^{l_j} \right)^{\omega_{i_j}} \right) \right)} \right)^{1/\sum_{i=1}^n l_i} \right\} \quad (96)$$

Hence, (91) is maintained. Thereafter,

$$0 \leq \left(\sqrt{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - (1 - \gamma_{i_j}^{2l_j})^{\omega_{i_j}} \right) \right)} \right)^{1/\sum_{i=1}^n l_i} \leq 1 \quad (97)$$

$$0 \leq \sqrt{1 - \left(1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - (1 - \eta_{i_j}^2)^{l_j} \right)^{\omega_{i_j}} \right) \right)}^{1/\sum_{i=1}^n l_i} \leq 1 \quad (98)$$

Because $\gamma^2 + \eta^2 \leq 1$,

$$\left(\left(\sqrt{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \gamma_{i_j}^{2l_j} \right)^{\omega_{i_j}} \right) \right)} \right)^{1/\sum_{j=1}^n l_j} \right)^2 + \left(\sqrt{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \eta_{i_j}^{2l_j} \right)^{\omega_{i_j}} \right) \right)} \right)^{1/\sum_{j=1}^n l_j} \right)^2 \leq \left(1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \gamma_{i_j}^{2l_j} \right)^{\omega_{i_j}} \right) \right) \right)^{1/\sum_{j=1}^n l_j} + \left(1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \eta_{i_j}^{2l_j} \right)^{\omega_{i_j}} \right) \right) \right)^{1/\sum_{j=1}^n l_j} = 1 \tag{99}$$

Property 13 (Idempotency) If $d_i (i = 1, 2, \dots, n)$ are equal, that is $d_i = d = (h, g)$, then

$$\text{DHPFDWBM}_{\omega}^l(d_1, d_2, \dots, d_n) = d \tag{100}$$

Proof

$$\begin{aligned} &\text{DHPFDWBM}_{\omega}^l(d_1, d_2, \dots, d_n) \\ &= \left(\bigoplus_{i_1, i_2, \dots, i_n=1}^n \left(\bigotimes_{j=1}^n \omega_{i_j} d^{l_j} \right) \right)^{1/\sum_{i=1}^n l_j} \\ &= \left(\bigoplus_{i_1, i_2, \dots, i_n=1}^n \left(\left(\bigotimes_{j=1}^n \omega_{i_j} \right) \cdot d^{\sum_{j=1}^n l_j} \right) \right)^{1/\sum_{i=1}^n l_j} \tag{101} \\ &= \left(\bigoplus_{i_1, i_2, \dots, i_n=1}^n \left(\bigotimes_{j=1}^n \omega_{i_j} \right) \right)^{1/\sum_{i=1}^n l_j} \cdot d \\ &= d \end{aligned}$$

Property 14 (Monotonicity) Let $d_j = (h_{d_j}, g_{d_j})$ and $b_j = (h_{b_j}, g_{b_j})$ ($j = 1, 2, \dots, n$) be two sets of DHPFNs.

If $\forall (\gamma_{d_j})^2 \leq \forall (\gamma_{b_j})^2, \gamma_{d_j} \in h_{d_j}, \gamma_{b_j} \in h_{b_j}$ and $\forall (\eta_{d_j})^2 \geq \forall (\eta_{b_j})^2, \eta_{d_j} \in g_{d_j}, \eta_{b_j} \in g_{b_j}$, then

$$\text{DHPFDWBM}_{\omega}^l(d_1, d_2, \dots, d_n) \leq \text{DHPFDWBM}_{\omega}^l(b_1, b_2, \dots, b_n) \tag{102}$$

Proof

$$\gamma_{d_j} \leq \gamma_{b_j} \tag{103}$$

$$\left(1 - \gamma_{d_j}^{2l_j} \right)^{\omega_{i_j}} \geq \left(1 - \gamma_{b_j}^{2l_j} \right)^{\omega_{i_j}} \tag{104}$$

$$\prod_{j=1}^n \left(1 - \left(1 - \gamma_{p_{i_j}}^{2l_j} \right)^{\omega_{i_j}} \right) \leq \prod_{j=1}^n \left(1 - \left(1 - \gamma_{b_{i_j}}^{2l_j} \right)^{\omega_{i_j}} \right) \tag{105}$$

Therefore,

$$\prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \gamma_{d_j}^{2l_j} \right)^{\omega_{i_j}} \right) \right) \geq \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \gamma_{b_j}^{2l_j} \right)^{\omega_{i_j}} \right) \right) \tag{106}$$

Thus,

$$\left(\sqrt{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \gamma_{d_j}^{2l_j} \right)^{\omega_{i_j}} \right) \right)} \right)^{1/\sum_{j=1}^n l_j} \leq \left(\sqrt{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \left(1 - \left(1 - \gamma_{b_j}^{2l_j} \right)^{\omega_{i_j}} \right) \right)} \right)^{1/\sum_{j=1}^n l_j} \tag{107}$$

which means $\forall \gamma_d^2 \leq \forall \gamma_b^2$; similarly, we can obtain $\forall \eta_d^2 \geq \forall \eta_b^2$.

If $\forall \gamma_{d_j}^2 < \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 > \forall \eta_{b_j}^2$, then

$$\begin{aligned} &\text{DHPFDWBM}_{\omega}^l(d_1, d_2, \dots, d_n) \\ &< \text{DHPFDWBM}_{\omega}^l(b_1, b_2, \dots, b_n); \end{aligned}$$

If $\forall \gamma_{d_j}^2 < \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 = \forall \eta_{b_j}^2$, then

$$\begin{aligned} &\text{DHPFDWBM}_{\omega}^l(d_1, d_2, \dots, d_n) \\ &< \text{DHPFDWBM}_{\omega}^l(b_1, b_2, \dots, b_n); \end{aligned}$$

If $\forall \gamma_{d_j}^2 = \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 > \forall \eta_{b_j}^2$, then

$$\begin{aligned} &\text{DHPFDWBM}_{\omega}^l(d_1, d_2, \dots, d_n) \\ &< \text{DHPFDWBM}_{\omega}^l(b_1, b_2, \dots, b_n); \end{aligned}$$

If $\forall \gamma_{d_j}^2 = \forall \gamma_{b_j}^2$ and $\forall \eta_{d_j}^2 = \forall \eta_{b_j}^2$, then

$$\begin{aligned} &\text{DHPFDWBM}_{\omega}^l(d_1, d_2, \dots, d_n) \\ &= \text{DHPFDWBM}_{\omega}^l(b_1, b_2, \dots, b_n); \end{aligned}$$

Therefore, the proof of Property 14 is completed.

Property 15 (Boundedness) Let $t, r > 0$ and $d_j = (h_{d_j}, g_{d_j}) (j = 1, 2, 3, \dots, n)$ be a collection of DHPFNs. If $d^+ = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{ \{ \max_i(\gamma_i) \}, \{ \min_i(\eta_i) \} \}$ and $d^- = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{ \{ \min_i(\gamma_i) \}, \{ \max_i(\eta_i) \} \}$, according to the property, there is

$$d^- \leq \text{DHPFDWBM}_{\omega}^l(d_1, d_2, \dots, d_n) \leq d^+ \tag{108}$$

Proof From Property 13, we can obtain the following:

$$\begin{aligned} & \text{DHPFDWBM}_\omega^l(d^+, d^+, \dots, d^+) \\ &= d^+, \text{DHPFDWBM}_\omega^l(d^-, d^-, \dots, d^-) = d^- \end{aligned} \tag{109}$$

From Property 14, we can obtain

$$\begin{aligned} d^- &= \text{DHPFDWBM}_\omega^l(d^-, d^-, \dots, d^-) \leq \\ & \text{DHPFDWBM}_\omega^l(d_1, d_2, \dots, d_n) \leq \\ & \text{DHPFDWBM}_\omega^l(d^+, d^+, \dots, d^+) = d^+ \end{aligned} \tag{110}$$

Therefore,

$$d^- \leq \text{DHPFDWBM}_\omega^l(d_1, d_2, \dots, d_n) \leq d^+ \tag{111}$$

DHPFDWGBM Operator

Thereafter, we extend DWGBM to DHPFN and introduce the dual hesitant Pythagorean fuzzy DWGBM (DHPFDWGBM) operator.

Definition 16 Let $l_j > 0$ and $d_j = (h_j, g_j) (j = 1, 2, \dots, n)$ be a set of DHPFN in which $h_p(x)$ and $g_p(x)$ are two sets of some values in $[0, 1]$ and weights vector being $W = (\omega_1, \omega_2, \dots, \omega_n)^T, \omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$. Then,

$$\begin{aligned} & \text{DHPFDWGBM}_\omega^l(d_1, d_2, \dots, d_n) \\ &= \frac{1}{\sum_{i=1}^n l_j} \left(\bigotimes_{i_1, i_2, \dots, i_n=1}^n \left(\bigoplus_{j=1}^n (l_j d_{i_j}) \right)^{\prod_{j=1}^n \omega_{i_j}} \right) \end{aligned} \tag{112}$$

Theorem 6 Let $l_j > 0$ and $d_j = (h_j, g_j) (j = 1, 2, \dots, n)$ be a collection of DHPFNs. The aggregated result of DHPFDWGBM is also a DHPFN

$$\begin{aligned} & \text{DHPFDWGBM}_\omega^l(d_1, d_2, \dots, d_n) = \frac{1}{\sum_{i=1}^n l_j} \left(\bigotimes_{i_1, i_2, \dots, i_n=1}^n \left(\bigoplus_{j=1}^n (l_j d_{i_j}) \right)^{\prod_{j=1}^n \omega_{i_j}} \right) \\ &= \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \sqrt{1 - \left(1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n (1 - \gamma_{i_j}^2)^{l_j} \right)^{\prod_{j=1}^n \omega_{i_j}} \right)^{1/\sum_{i=1}^n l_j}} \right\} \right. \\ & \left. \left\{ \left(\sqrt{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \eta_{i_j}^{2l_j} \right)^{\prod_{j=1}^n \omega_{i_j}} \right)^{1/\sum_{i=1}^n l_j}} \right\} \right\} \end{aligned} \tag{113}$$

Proof Through Definition 4, we can obtain

$$l_j d_{i_j} = \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \sqrt{1 - (1 - \gamma_{i_j}^2)^{l_j}} \right\}, \left\{ \eta_{i_j}^{l_j} \right\} \right\} \tag{114}$$

Thereafter,

$$\bigoplus_{j=1}^n l_j d_{i_j} = \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \sqrt{1 - \prod_{j=1}^n (1 - \gamma_{i_j}^2)^{l_j}} \right\}, \left\{ \prod_{j=1}^n \eta_{i_j}^{l_j} \right\} \right\} \tag{115}$$

And,

$$\begin{aligned} & \left(\bigoplus_{j=1}^n l_j d_{i_j} \right)^{\prod_{j=1}^n \omega_{i_j}} \\ &= \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \left(\sqrt{1 - \prod_{j=1}^n (1 - \gamma_{i_j}^2)^{l_j}} \right)^{\prod_{j=1}^n \omega_{i_j}} \right\}, \right. \\ & \left. \left\{ \sqrt{1 - \left(1 - \prod_{j=1}^n \eta_{i_j}^{2l_j} \right)^{\prod_{j=1}^n \omega_{i_j}}} \right\} \right\} \end{aligned} \tag{116}$$

Therefore,

$$\begin{aligned} & \bigotimes_{i_1, i_2, \dots, i_n=1}^n \left(\bigoplus_{j=1}^n l_j d_{i_j} \right)^{\prod_{j=1}^n \omega_{i_j}} = \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \prod_{i_1, i_2, \dots, i_n=1}^n \left(\sqrt{1 - \prod_{j=1}^n (1 - \gamma_{i_j}^2)^{l_j}} \right)^{\prod_{j=1}^n \omega_{i_j}} \right\}, \right. \\ & \left. \left\{ \sqrt{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \eta_{i_j}^{2l_j} \right)^{\prod_{j=1}^n \omega_{i_j}}} \right\} \right\} \end{aligned} \tag{117}$$

Thus,

$$\begin{aligned} & \text{DHPFDWGBM}_\omega^l(d_1, d_2, \dots, d_n) = \frac{1}{\sum_{i=1}^n l_j} \left(\bigotimes_{i_1, i_2, \dots, i_n=1}^n \left(\bigoplus_{j=1}^n (l_j d_{i_j}) \right)^{\prod_{j=1}^n \omega_{i_j}} \right) \\ &= \cup_{\gamma_{i_j} \in h_{i_j}, \eta_{i_j} \in g_{i_j}} \left\{ \left\{ \sqrt{1 - \left(1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n (1 - \gamma_{i_j}^2)^{l_j} \right)^{\prod_{j=1}^n \omega_{i_j}} \right)^{1/\sum_{i=1}^n l_j}} \right\}, \right. \\ & \left. \left\{ \left(\sqrt{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \eta_{i_j}^{2l_j} \right)^{\prod_{j=1}^n \omega_{i_j}} \right)^{1/\sum_{i=1}^n l_j}} \right\} \right\} \end{aligned} \tag{118}$$

Hence, (113) is maintained. Thereafter,

$$1 \leq \sqrt[n]{1 - \left(1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n (1 - \gamma_{i_j}^2)^{l_j}\right)^{\prod_{j=1}^n \omega_{i_j}}\right)^{1/\sum_{i=1}^n l_i}} \leq 0 \tag{119}$$

$$1 \leq \left(\sqrt[n]{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \eta_{i_j}^{2l_j}\right)^{\prod_{j=1}^n \omega_{i_j}}}\right)^{1/\sum_{i=1}^n l_i} \leq 0 \tag{120}$$

Because of $\gamma^2 + \eta^2 \leq 1$,

$$\left(\sqrt[n]{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \eta_{i_j}^{2l_j}\right)^{\prod_{j=1}^n \omega_{i_j}}}\right)^{1/\sum_{i=1}^n l_i} \tag{121}$$

$$\leq \sqrt[n]{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n (1 - \gamma_{i_j}^2)^{l_j}\right)^{\prod_{j=1}^n \omega_{i_j}}} \tag{121}$$

Therefore,

$$\left(\sqrt[n]{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n (1 - \gamma_{i_j}^2)^{l_j}\right)^{\prod_{j=1}^n \omega_{i_j}}}\right)^2 + \left(\sqrt[n]{1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n \eta_{i_j}^{2l_j}\right)^{\prod_{j=1}^n \omega_{i_j}}}\right)^2 \leq \tag{122}$$

$$1 - \left(1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n (1 - \gamma_{i_j}^2)^{l_j}\right)^{\prod_{j=1}^n \omega_{i_j}}\right)^{1/\sum_{i=1}^n l_i} + \left(1 - \prod_{i_1, i_2, \dots, i_n=1}^n \left(1 - \prod_{j=1}^n (1 - \gamma_{i_j}^2)^{l_j}\right)^{\prod_{j=1}^n \omega_{i_j}}\right)^{1/\sum_{i=1}^n l_i} = 1$$

thereby completing the proof.

Similar to DHPFDWBM, the DHPFDWGBM has the same properties. The proofs of these properties are similar to that of the properties of DHPFDWGBM. Accordingly, the proofs are omitted to save space.

Property 16 (Idempotency) If $d_i (i = 1, 2, \dots, n)$ are equal, that is $d_i = d = (h, g)$, then

$$\text{DHPFDWGBM}_{\omega}^l(d_1, d_2, \dots, d_n) = d \tag{123}$$

Property 17 (Monotonicity) Let $d_j = (h_{d_j}, g_{d_j})$ and $b_j = (h_{b_j}, g_{b_j})$ ($j = 1, 2, 3, \dots, n$) be two sets of DHPFNs.

If $\forall (\gamma_{d_j})^2 \leq \forall (\gamma_{b_j})^2, \gamma_{d_j} \in h_{d_j}, \gamma_{b_j} \in h_{b_j}$ and $\forall (\eta_{d_j})^2 \geq \forall (\eta_{b_j})^2, \eta_{d_j} \in g_{d_j}, \eta_{b_j} \in g_{b_j}$, then

$$\text{DHPFDWGBM}_{\omega}^l(d_1, d_2, \dots, d_n) \leq \text{DHPFDWGBM}_{\omega}^l(b_1, b_2, \dots, b_n) \tag{124}$$

Property 18 (Boundedness) If $d^+ = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\{\max_i(\gamma_i)\}, \{\min_i(\eta_i)\}\}$ and $d^- = \cup_{\gamma_j \in h_{d_j}, \eta_j \in g_{d_j}} \{\{\min_i(\gamma_i)\}, \{\max_i(\eta_i)\}\}$, according to the property, there is

$$d^- \leq \text{DHPFDWGBM}_{\omega}^l(d_1, d_2, \dots, d_n) \leq d^+ \tag{125}$$

Methods for MADM with DHPFNs

Based on the DHPFWBM and DHPFWGBM operators, we shall propose the model for MADM with DHPFNs. Let $A = \{A_1, A_2, \dots, A_m\}$ be a discrete set of alternatives and $G = \{G_1, G_2, \dots, G_n\}$ be the set of attributes, $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the weighting vector of the attribute $G_j (j = 1, 2, \dots, n)$, where $\omega_j \in [0, 1], \sum_{j=1}^n \omega_j = 1$.

Suppose that $d = (d_{cj})_{m \times n} = (h_{cj}, g_{cj})_{m \times n}$ is the Pythagorean fuzzy decision matrix, where h_{cj} set indicates the degree that the alternative A_c satisfies the attribute G_j given by the decision maker, g_{cj} set indicates the degree that the alternative A_c does not satisfy the attribute G_j given by the decision maker, $\gamma_{cj} \in h_{cj} \subset [0, 1], \eta_{cj} \in g_{cj} \subset [0, 1], (\gamma_{cj})^2 + (\eta_{cj})^2 \leq 1, c = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

In the following, we apply the DHPFWBM (DHPFWGBM) operator to the MADM problems with DHPFNs.

Step 1. We utilize the DHPFNs given in matrix \tilde{R} , and the DHPFWBM operator

$$\begin{aligned} d_c &= \text{DHPFWBM}_{\omega}^{l,r}(d_{c1}, d_{c2}, \dots, d_{cn}) = \left(\bigoplus_{i,j=1}^n \omega_i \omega_j (d_{ci}^l \otimes d_{cj}^r)\right)^{1/(t+r)} \\ &= \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt[n]{1 - \prod_{i,j=1}^n \left(1 - \gamma_{ci}^{2t} \gamma_{cj}^{2r}\right)^{\omega_i \omega_j}}\right)^{1/(t+r)} \right\}, \right. \\ &\quad \left. \left\{ \sqrt[n]{1 - \prod_{i,j=1}^n \left(1 - (1 - \eta_{ci}^2)^t (1 - \eta_{cj}^2)^r\right)^{\omega_i \omega_j}}\right)^{1/(t+r)} \right\} \right\} \\ &c = 1, 2, \dots, m. \end{aligned} \tag{126}$$

or

$$d_c = \text{DHPFWGBM}_\omega^{t,r}(d_{c1}, d_{c2}, \dots, d_{cn})$$

$$= \cup_{\gamma \in h, \eta \in g} \left\{ \left\{ \left(\sqrt{1 - \left(1 - \prod_{i,j=1}^n \left(1 - (1 - \gamma_{ci}^2)^t (1 - \gamma_{cj}^2)^r \right)^{\omega_i \omega_j} \right)} \right)^{1/(t+r)} \right\}, \left\{ \left(\sqrt{1 - \prod_{i,j=1}^n \left(1 - \eta_{ci}^{2t} \eta_{cj}^{2r} \right)^{\omega_i \omega_j} \right)} \right)^{1/(t+r)} \right\} \right\} c = 1, 2, \dots, m. \tag{127}$$

to derive the $d_c (c = 1, 2, \dots, m)$ of the alternative A_c .

- Step 2. Calculate the scores $S(d_c) (c = 1, 2, \dots, m)$ of the overall DHPFNs $d_c (c = 1, 2, \dots, m)$ to rank all the alternatives $A_c (c = 1, 2, \dots, m)$ and then to select the best one(s). If there is no difference between two scores $S(d_c)$ and $S(d_s)$, then we need to calculate the accuracy degrees $H(d_c)$ and $H(d_s)$ of the overall DHPFNs d_c and d_s , respectively, and then rank the alternatives A_c and A_s in accordance with the accuracy degrees $H(d_c)$ and $H(d_s)$.
- Step 3. Rank all the alternatives $A_c (c = 1, 2, \dots, m)$ and select the best one(s) in accordance with $S(d_c) (c = 1, 2, \dots, m)$.
- Step 4. End.

that the government will seriously treat the environmental pollution problems. In order to control the total energy consumption, the manufacturing sector is one of the key targets of strengthening energy conservation. Faced with government regulation, legal constraints, and public opinion pressure, manufacturing enterprises have to take necessary measures to achieve “Green Development”. One of these measures is to implement green supply chain management. GSCM emphasis on the joint development of the socioeconomic and ecological environment, requiring companies to consider environmental factors in product design, packaging, procurement, production, sales, logistics, waste, recycling and other aspects. Supplier is the “Source” of the whole supply chain, and the green supplier selection is the foundation of green supply chain management. The quality of suppliers will directly affect the environmental performance of enterprises. First, the green supply chain management and the traditional supply chain management were compared. Then, the characteristics of green supplier partnerships were analyzed from various aspects. Then, we shall give an application to select green suppliers in GSCM with DHPFNs. There are five possible green suppliers in GSCM $A_i (i = 1, 2, 3, 4, 5)$ to select. The experts select four attribute to assess the five possible green suppliers: (1) G_1 is the product quality factor, (2) G_2 is environmental factors, (3) G_3 is delivery factor, and (4) G_4 is price factor. Five green suppliers $A_i (i = 1, 2, 3, 4, 5)$ are to be evaluated with DHPFNs according to four attributes (whose weighting

Case Study and Comparative Analysis

Case Study

Nowadays, the ecological environment around the world is deteriorating and human survival is threatened. As a result of it, the development of circular economy has become a hot topic. In 2015, the government proposed to actively promote economic restructuring and upgrading, and the manufacturing sector is the focus of rectification and reform. In the same year, our country also implemented a new environmental law. It means

Table 1 DHPFN decision matrix

	G_1	G_2	G_3	G_4
A_1	{(0.4,0.2), (0.4,0.3)}	{(0.4,0.6), (0.5,0.4), (0.6,0.1)}	{(0.3,0.5), (0.4,0.3)}	{(0.6,0.3), (0.7,0.4)}
A_2	{(0.5,0.1), (0.6,0.2)}	{(0.5,0.6), (0.6,0.4)}	{(0.5,0.3), (0.5,0.4)}	{(0.4,0.5), (0.5,0.2), (0.6,0.4)}
A_3	{(0.5,0.4), (0.7,0.4)}	{(0.6,0.3), (0.6,0.5)}	{(0.5,0.2), (0.6,0.4), (0.7,0.2)}	{(0.5,0.3), (0.6,0.4)}
A_4	{(0.5,0.3), (0.5,0.4)}	{(0.6,0.5), (0.6,0.7)}	{(0.4,0.3), (0.5,0.3)}	{(0.4,0.6), (0.5,0.4)}
A_5	{(0.5,0.3), (0.6,0.5)}	{(0.6,0.4), (0.6,0.5)}	{(0.5,0.4), (0.6,0.4)}	{(0.2,0.3), (0.3,0.4)}

Table 2 The rank and score of green suppliers by using DHPF operators

	A_1	A_2	A_3	A_4	A_5	Order
DHPFWBM	0.6012	0.5810	0.6104	0.5336	0.5702	$A_3 > A_1 > A_2 > A_5 > A_4$
DHPFWGBM	0.5362	0.5327	0.5883	0.4785	0.5154	$A_3 > A_1 > A_2 > A_5 > A_4$
DHPFGWBM	0.5987	0.5763	0.6094	0.5310	0.5695	$A_3 > A_1 > A_2 > A_5 > A_4$
DHPFGWGBM	0.2264	0.2349	0.2547	0.2307	0.2176	$A_3 > A_2 > A_4 > A_1 > A_5$
DHPFDWBM	0.5974	0.5743	0.6090	0.5300	0.5691	$A_3 > A_1 > A_2 > A_5 > A_4$
DHPFDWGBM	0.2040	0.2131	0.2349	0.2085	0.1949	$A_3 > A_2 > A_4 > A_1 > A_5$

According to the result of green suppliers' order, we can know that the best choice is supplier A_3 ; we get the same result by different aggregations, which proved the effectiveness of result

vectors $\omega = (0.2, 0.4, 0.1, 0.3), t = r = 3, s = t = r = 3, l_i = 3, i = 1, \dots, 4$, are as shown in Table 1.

Step 1 We utilize the DHPFNs given in matrix \tilde{R} and the DHPFWBM operator to get aggregating result. We illustrate one of alternative for save space.

Step 2 The aggregating results and the score functions of the green suppliers are shown in Table 2.

$$\begin{aligned}
 d_1 &= \text{DHPFWBM}_{\omega}^{t,r}(G_1, G_2, G_3, G_4) \\
 &= \left\{ \{(0.4, 0.2), (0.4, 0.3)\}, \{(0.4, 0.6), (0.5, 0.4), (0.6, 0.1)\}, \right. \\
 &\quad \left. \{(0.3, 0.5), (0.4, 0.3)\}, \{(0.6, 0.3), (0.7, 0.4)\} \right\} \\
 &= \left\{ (0.5047, 0.4016), (0.5796, 0.4364), (0.5064, 0.3809), (0.5805, 0.4148), \right. \\
 &\quad (0.5259, 0.3367), (0.5908, 0.3676), (0.5272, 0.3182), (0.5916, 0.3484), \\
 &\quad (0.5679, 0.2103), (0.6165, 0.2350), (0.5689, 0.1965), (0.6172, 0.2203), \\
 &\quad (0.5047, 0.4256), (0.5796, 0.4600), (0.5064, 0.4045), (0.5805, 0.4383), \\
 &\quad (0.5259, 0.3578), (0.5908, 0.3887), (0.5272, 0.3390), (0.5916, 0.3693), \\
 &\quad \left. (0.5679, 0.2310), (0.6165, 0.2570), (0.5689, 0.2163), (0.6172, 0.2414) \right\}
 \end{aligned}$$

Influence of the Parameter on the Final Result

The aggregation method of extend DHPFS with BM operator has two advantages, one is that it can reduce the bad effects of the unduly high and low assessments on the final result, and the other is that it can capture the interrelationship between dual hesitate Pythagorean fuzzy numbers. These aggregation operators have a parameter vector, which make extended operator more flexible, so the different vectors lead to different aggregation results, different scores, and ranking results. In order to illustrate the influence of the parameter vector l_i on

Table 3 Ranking results by utilizing different parameter vector l_i in the DHPFDWBM operator

$l_i, i = 1, \dots, 6$	Scores					Order
	A_1	A_2	A_3	A_4	A_5	
(1,1,1,1)	0.5714	0.5633	0.6014	0.5107	0.5437	$A_3 > A_1 > A_2 > A_5 > A_4$
(2,2,2,2)	0.5853	0.5688	0.6052	0.5205	0.5598	$A_3 > A_1 > A_2 > A_5 > A_4$
(3,3,3,3)	0.5974	0.5743	0.6090	0.5300	0.5691	$A_3 > A_1 > A_2 > A_5 > A_4$
(4,4,4,4)	0.6074	0.5795	0.6128	0.5387	0.5752	$A_3 > A_1 > A_2 > A_5 > A_4$
(5,5,5,5)	0.6153	0.5844	0.6165	0.5464	0.5796	$A_3 > A_1 > A_2 > A_5 > A_4$
(6,6,6,6)	0.6218	0.5872	0.6199	0.5531	0.5830	$A_1 > A_3 > A_2 > A_5 > A_4$

Table 4 Ranking results by utilizing different parameter vector l_i in the DHPFDWGBM operator

$l_i, i = 1, \dots, 6$	Scores					Order
	O_1	O_2	O_3	O_4	O_5	
(1,1,1,1)	0.3268	0.3293	0.3424	0.3271	0.3223	$A_3 > A_2 > A_4 > A_1 > A_5$
(2,2,2,2)	0.2402	0.2461	0.2653	0.2425	0.2331	$A_3 > A_2 > A_4 > A_1 > A_5$
(3,3,3,3)	0.2040	0.2131	0.2349	0.2085	0.1949	$A_3 > A_2 > A_4 > A_1 > A_5$
(4,4,4,4)	0.1830	0.1951	0.2182	0.1897	0.1720	$A_3 > A_2 > A_4 > A_1 > A_5$
(5,5,5,5)	0.1687	0.1836	0.2073	0.1774	0.1559	$A_3 > A_2 > A_4 > A_1 > A_5$
(6,6,6,6)	0.1580	0.1754	0.1996	0.1685	0.1434	$A_3 > A_2 > A_4 > A_1 > A_5$

Table 5 The aggregating result of existing method

Method name	Scores					Order
	A_1	A_2	A_3	A_4	A_5	
DHPFWA	0.5651	0.5874	0.6206	0.5440	0.5603	$A_3 > A_2 > A_1 > A_5 > A_4$
DHPFWG	0.5284	0.5582	0.6056	0.5109	0.5385	$A_3 > A_2 > A_5 > A_1 > A_4$

the ranking result, we discuss the influence with several parameter vectors; you can find the results in Tables 3 and 4.

We can see that the different parameters lead to different results and different ranking orders. The more attributes we consider, the bigger the scores; the bigger the attribute value, the lower the scores. Therefore, the parameter vector can be considered as decision maker's risk preference.

Comparative Analysis

In this section, we get aggregation result by some existing methods, such as DHPFWA operator and DHPFWG operator [42], and we can find the results in Table 5.

From the scores and order result overall, we can see that these result differently with the novel method we proposed; it is because that these methods in Table 5 do not consider the interrelationship of DHPFNs.

Conclusion

In this paper, we investigate the MADM problems based on the BM operators with DHPFNs. Firstly, we introduce the concept and basic operations of the DHPFNs. Then, motivated by the idea of Bonferroni mean operators, we have developed some BM operators for aggregating the DHPFNs. The main characteristics of these proposed operators are also analyzed. Then, we have used these operators to solve the MADM problems with DHPFNs. Finally, a practical example for supplier selection in GSCM is given to verify the developed approach and comparative analysis is analyzed with existing models. In the future, we shall continue studying the MADM problems with the application and extension of the developed operators to other fuzzy environments [22, 59–70] and uncertain MADM problems [71–78].

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Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors.

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