Cognitive Computation (2019) 11:125–144
https://doi.org/10.1007/s12559-018-9597-2

A New Decision-Making Method Based on Interval-Valued Linguistic Intuitionistic Fuzzy Information

Peide Liu [1] · Xiyou Qin [1]

Received: 18 March 2017 / Accepted: 19 September 2018 / Published online: 31 October 2018

Abstract

In real decision-making, because of the particularity of human cognition activity, it is difficult to depict the decision information by exact numbers, especially, for complex decision information, how to express and aggregate them is an important work for solving these decision-making problems. In order to express the complex fuzzy information accurately, we proposed the concept of interval-valued linguistic intuitionistic fuzzy numbers (IVLIFNs), where their membership function and non-membership function are represented by interval-valued linguistic terms, and then we developed the operational rules, score function, accuracy function, and comparison method of them. Considering that the Maclaurin symmetric mean (MSM) operator has a good characteristic in dealing with the interrelationships among multi-parameters, and it also is a generalization of arithmetic aggregation operator, Bonferroni mean (BM) operator, and geometric aggregation operator, we further proposed the interval-valued linguistic intuitionistic fuzzy MSM (IVLIFMSM) operator, the weighted interval-valued linguistic intuitionistic fuzzy MSM (WIVLIFMSM) operator, and proved some related properties of them. We gave an illustrative example to demonstrate the steps and the effectiveness of the proposed method by the comparison with existing methods. IVLIFNs can more conveniently express the complex fuzzy information in qualitative environment by considering the cognition of decision-makers, and the proposed method can consider the interrelationship among multiple input arguments, so it can make the decision-making results more reasonable. In a word, the proposed method is more scientific and flexible in solving multiple attribute decision-making (MADM) problems than some existing methods.

Keywords Multiple attribute decision-making · Interval-valued linguistic intuitionistic fuzzy numbers · Maclaurin symmetric mean

Introduction

Multiple attribute decision-making (MADM) problems have gotten more and more attentions by researchers. Because of the particularity of human cognition activity, how to depict the complex fuzzy information and make the decision more scientific and accurate is an important work to solve the MADM problems [13, 36, 48]. Since the concept of fuzzy set (FS) was proposed by Zadeh [45], the fuzzy MADM has been a hot research topic, and a lot of achievements have been made. Because the FS only has one membership function, sometimes, it is difficult to express the fuzzy information efficiently, such as the voting problem in which there exist some oppositions. In order to solve this problem, Atanassov [2, 3] proposed some new concepts called intuitionistic fuzzy set (IFS) and intuitionistic fuzzy numbers (IFNs), in which the non-membership function is added. However, in IFNs, both membership function and non-membership function can only be depicted by crisp numbers, and sometimes, it is still difficult to express the decision-making information due to the complexity of decision-makers' cognition [37]. Further, Atanassov [4], Atanassov and Gargov [5] extended the membership function and non-membership function of IFS to interval numbers and defined the concept and some operational rules of interval-valued IFNs (IVIFNs).

✉ Peide Liu
peide.liu@gmail.com

[1] School of Management Science and Engineering, Shandong University of Finance and Economics, Jinan 250014, Shandong, China

Now, the IVIFNs have gotten lots of research achievements, and Chen et al. [9] proposed one new ranking method for IVIFNs. Based on IVIFNs, some researchers proposed a lot of relevant aggregation operators for IVIFNs [18–29, 41, 44, 47]. For example, Yu et al. [44] developed some prioritized aggregation operators for IVIFNs; Zhao et al. [47] proposed some new generalized operators with arithmetic mean and geometric mean for IVIFNs, respectively.

However, in real decision-making, because of the particularity of human cognition activity [43], decision-makers cannot give the evaluation by the numerical value but can give the fuzzy qualitative information easily. So, it is still necessary to give the quantitative value for membership function and non-membership function in IVIFNs. For example, about the evaluation on the quality for a car, people can give a result such as "good," "very good," "bad," or "very bad" easily. Of course, for some attributes, we may get an exact value by some scientific methods on the basis of spending a great deal of cost; however, people can only give the qualitative value because of the particularity of human cognition activity. For example, about the speed of a car in highway, we can measure it by GPS and give an exact speed "130 km/h"; however, people can only give the result such as "very fast" and "fast." Generally, in a quantitative setting, we use the numerical values to express the information and can get an effective result. In a qualitative setting, it is very feasible by linguistic terms instead of numerical values [16, 46]. Considering the decision maker's cognition, in order to easily express the membership function and non-membership function in a qualitative setting, Chen and Liu [10] proposed the linguistic IFNs (LIFNs), in which the membership function and non-membership function are expressed by linguistic variables. LIFNs can more easily deal with the fuzzy information in quantitative setting and can combine the advantages of both IFNs and linguistic variables.

IVIFNs are more convenient to express the imprecise fuzzy information than IFNs; however, their membership function and non-membership function are expressed by interval numbers. Similarly, in qualitative setting, considering the cognition of decision-makers is more subjective, it is flexible to express the membership function and non-membership function by interval-valued linguistic variables than interval numbers. So, one of our goals in this paper is to propose the interval-valued LIFNs (IVLIFNs), in which the membership function and non-membership function are presented by interval-valued linguistic variables, and then to establish their operational rules and comparison method.

In addition, in actual MADM problems, sometimes, it is very important to consider the interrelationship among multi-attributes; there are some aggregation operators which can capture the interrelationship among the aggregated attributes, such as BM [8, 21, 23, 24, 30] and the Heronian mean (HM) [17, 25, 27], which consider interrelationships between two aggregated arguments. However, maybe we need to consider the interrelationships among any number of aggregated arguments. In order to achieve this function, Maclaurin [31] proposed Maclaurin symmetric mean (MSM) which is an important aggregation operator with capturing the interrelationships among multiple input arguments. Because it has one adjustable parameter, this operator can simplify to some other existing operators, such as arithmetic aggregation operator, BM operator, and geometric aggregation operator. Compared with some aggregation operators which can capture the interrelationship among the aggregated attributes, the MSM operator is more flexible and effective. In the past years, MSM was developed not only in theories but also in applications of inequality fields [12], and a lot of research results based on MSM [1, 6, 14, 33] were obtained. Qin and Liu [33, 34] developed MSM operator to aggregate information under multiple-attribute fuzzy environment. Qin and Liu [35] proposed the conception of the dual Maclaurin symmetric mean (DMSM) operator on the basis of MSM. Although MSM has the significant advantages in capturing the interrelationships among any number of input arguments, now, it cannot process the IVLIFNs. Obviously, it is necessary to extend the MSM to process the IVLIFNs. So, the second goal of this paper is to propose some MSM operators for IVLIFNs, and then use them to solve MADM problems with IVLIFNs.

The remainder of the paper is arranged as follows: In part 2, we review some basic concepts of IVIFNs and LIFNs. In part 3, we propose the notion of IVLIFNs and give the operational laws and their properties. In part 4, we propose a MSM operator for IVLIFNs and proved some properties of this operator. In part 5, we develop the MADM method with IVLIFNs. In part 6, we give one example and compare with some existing MADM method. In part 7, we give summary and further research of this paper.

Preliminaries

The LIFNs

In some qualitative environments, in order to conveniently express fuzzy information, linguistic variables are adopted because they are a good tool to depict the qualitative information [11, 15, 39, 40].

Suppose that $S = \{s_i | i = 0, 1, ..., \tau\}$ is a linguistic term set (LTS), where $\tau + 1$ is an odd cardinality, s_i is one possible value from LTS S, and we call it a linguistic variable. The number and semantics of LTS are determined according to real decision-making problems. For instance, S is one LTS with nine terms for evaluating the production quality, which is given as follows [10]:

$$S = \{s_0 = \text{extremely bad}, s_1 = \text{very bad}, s_2 = \text{bad}, s_3 = \text{slightly bad}, s_4 = \text{fair}, s_5 = \text{slightly good}, s_6 = \text{good}, s_7 = \text{very good}, s_8 = \text{extremely good}\}$$

For convenience, we use the $S_{[0,\tau]}$ to express the LTS $S = \{s_i | i = 0, 1, ..., \tau\}$.

About the operational rules and properties of linguistic variables, please refer to references [11, 15].

Definition 1 [10]. Let $s_m, s_n \in S_{[0,\tau]}$ and $\gamma = (s_m, s_n)$, where $m + n \in [0, \tau]$, then we call γ a LIFN.

Definition 2 [10]. If $\gamma_1 = (s_{m_1}, s_{n_1})$ and $\gamma_2 = (s_{m_2}, s_{n_2}) \in S_{[0,\tau]}$ are two LIFNs, then

$$\gamma_1 \oplus \gamma_2 = (s_{m1}, s_{n1}) \oplus (s_{m2}, s_{m2}) = \left(s_{m1+m2-\frac{m1m2}{\tau}}, s_{\frac{n1n2}{\tau}}\right); \quad (1)$$

$$\gamma_1 \otimes \gamma_2 = (s_{m1}, s_{n1}) \otimes (s_{m2}, s_{n2}) = \left(s_{\frac{m1m2}{\tau}}, s_{n1+n2-\frac{n1n2}{\tau}}\right); \quad (2)$$

$$\lambda\gamma = \lambda(s_m, s_n) = \left(s_{\tau-\tau\left(1-\frac{m}{\tau}\right)^{\lambda}}, s_{\tau\left(\frac{n}{\tau}\right)^{\lambda}}\right); \quad (3)$$

$$\gamma^{\lambda} = (s_m, s_n)^{\lambda} = \left(s_{\tau\left(\frac{m}{\tau}\right)^{\lambda}}, s_{\tau-\tau\left(1-\frac{n}{\tau}\right)^{\lambda}}\right). \quad (4)$$

Definition 3 [10]. If $\gamma = (s_m, s_n)$ is a LIFN, the score value and the accuracy value are defined as follows:

$$(1)\ \textit{the score value is } Ls(\gamma) = m - n; \quad (5)$$

$$(2)\ \textit{the accuracy value is } Lh(\gamma) = m + n. \quad (6)$$

where $Ls(\gamma) \in [-\tau, \tau]$ and $Lh(\gamma) \in [0, \tau]$.

Definition 4 [10] Let $\gamma_1 = (s_{m_1}, s_{n_1})$ and $\gamma_2 = (s_{m_2}, s_{n_2})$ be two LIFNs, $Ls(\gamma_1)$ and $Ls(\gamma_2)$ be the score values of γ_1 and γ_2, respectively, $Lh(\gamma_1)$ and $Lh(\gamma_2)$ be the accuracy values of γ_1 and γ_2, respectively. Then,

(1) If $Ls(\gamma_1) > Ls(\gamma_2)$, we get $\gamma_1 > \gamma_2$;
(2) If $Ls(\gamma_1) = Ls(\gamma_2)$, we get

If $Lh(\gamma_1) > Lh(\gamma_2)$, we get $\gamma_1 > \gamma_2$;
If $Lh(\gamma_1) = Lh(\gamma_2)$, we get $\gamma_1 = \gamma_2$.

The IVIFNs

Definition 5 [3, 4]. Suppose $\hat{X} = \{\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\}$ is a universe of discourse, an IVIFS $\hat{D}$ in $\hat{X}$ can be expressed by

$$\hat{D} = \left\{ < \hat{x}, \hat{u}_{\hat{D}}\left(\hat{x}\right), \hat{v}_{\hat{D}}\left(\hat{x}\right) > \hat{x} \in \hat{X} \right\} \quad (7)$$

where $\hat{u}_{\hat{D}}(\hat{x}) \subseteq [0, 1]$ and $\hat{v}_{\hat{D}}(\hat{x}) \subseteq [0, 1]$ are interval numbers, with the condition $0 \le \sup\left(\hat{u}_{\hat{D}}(\hat{x})\right) + \sup\left(\hat{v}_{\hat{D}}(\hat{x})\right) \le 1$, $\forall \hat{x} \in \hat{X}$. The numbers $\hat{u}_{\hat{D}}(\hat{x})$ and $\hat{v}_{\hat{D}}(\hat{x})$ represent the membership function and non-membership function of the element x to the set $\hat{D}$ respectively. For convenience, let $\hat{u}_{\hat{D}}(\hat{x}_i) = [e, f]$, $\hat{v}_{\hat{D}}(\hat{x}_i) = [g, h]$, then $\hat{d} = ([e, f], [g, h])$ is called an IVIFN.

Definition 6 [38]. If $\hat{d} = ([e, f], [g, h])$ and $\hat{d}_0 = ([e_0, f_0], [g_0, h_0])$ are two IVIFNs, and $\lambda > 0$, then the operational rules are defined as follows:

$$(1)\quad \hat{d} + \hat{d}_0 = ([e + e_0 - e\, e_0, f + f_0 - f\, f_0], [g\, g_0, h\, h_0]); \quad (8)$$

$$(2)\quad \hat{d} \times \hat{d}_0 = ([e\, e_0, f\, f_0], [g + g_0 - g\, g_0, h + h_0 - h\, h_0]); \quad (9)$$

$$(3)\quad \lambda\hat{d} = \left(\left[1-(1-e)^{\lambda}, 1-(1-f)^{\lambda}\right], \left[g^{\lambda}, h^{\lambda}\right]\right); \quad (10)$$

$$(4)\quad \hat{d}^{\lambda} = \left(\left[e^{\lambda}, f^{\lambda}\right], \left[1-(1-g)^{\lambda}, 1-(1-h)^{\lambda}\right]\right). \quad (11)$$

Definition 7 [9]. Let $\hat{d} = ([e, f], [g, h])$ be an IVIFN; the score value sf of the IVIFN $\hat{d}$ can be represented as follows:

$$sf\left(\hat{d}\right) = \frac{e + f - g - h}{2} \quad (12)$$

Definition 8 [9]. Let $\hat{d} = ([e, f], [g, h])$ be an IVIFN; the accuracy value hf of the IVIFN $\hat{d}$ can be represented as follows:

$$hf\left(\hat{d}\right) = \frac{e + f + g + h}{2} \quad (13)$$

Definition 9 [9]. If $\hat{d}=([e,f],[g,h])$ and $\hat{d}_1=([e_1,f_1],[g_1,h_1])$ are any two IVIFNs, then we have

(1) If $sf(\hat{d}_1)>sf(\hat{d}_2)$, then $\hat{d}_1>\hat{d}_2$;
(2) If $sf(\hat{d}_1)=sf(\hat{d}_2)$, then

If $hf(\hat{d}_1)>hf(\hat{d}_2)$, then $\hat{d}_1>\hat{d}_2$;
If $hf(\hat{d}_1)=hf(\hat{d}_2)$, then $\hat{d}_1=\hat{d}_2$.

The MSM Operator

Definition 10 [31]. Let $(\beta_1, \beta_2, ..., \beta_n)$ be a group of positive numbers, and $x = 1, 2, 3, ..., n$. If

$$MSM^{(x)}(\beta_1,\beta_2,\beta_3,...,\beta_n)=\left(\frac{\underset{1\le i_1<...<i_x\le n}{\oplus}\overset{x}{\underset{j=1}{\otimes}}\beta_{i_j}}{C_n^x}\right)^{\frac{1}{x}} \quad (14)$$

Then, we call $MSM^{(x)}$ as the MSM operator, where C_n^x is the coefficient of binomial and $(i_1, i_2, ..., i_n)$ includes all the x-tuple combination of $(1, 2, 3, ..., n)$.

The MSM operator has the following properties:

(1) $MSM^{(x)}(0,0,0,...,0)=0$ (15)

(2) $MSM^{(x)}(\beta,\beta,\beta,...,\beta)=\beta$ (16)

(3) $MSM^{(x)}(\beta_1,\beta_2,...,\beta_n)\le MSM^{(x)}(\delta_1,\delta_2,...,\delta_n)$ (17)

If $\beta_i \le \delta_i$ for all i, $i = 1, 2, ..., n$;

(4) $\min\{\beta_i\}\le MSM^{(x)}(\beta_1,\beta_2,...,\beta_n)\le\max\{\beta_i\}$. (18)

The IVLIFNs

In this part, we propose the concept of IVLIFNs in which the membership function and non-membership function are presented by interval-valued linguistic variables, and we also discuss some properties about IVLIFNs.

Definition 11 Let $\hat{\gamma}=(s_{\hat{\alpha}},s_{\hat{\beta}})$, where $\hat{\alpha}=[k,l]\subseteq[0,\tau]$, $\hat{\beta}=[p,q]\subseteq[0,\tau]$ are interval numbers, if $0\le l+q\le\tau$, then the $\hat{\gamma}$ can call an IVLIFN, where $s_{\hat{\alpha}}$ expresses the membership function and $s_{\hat{\beta}}$ expresses the non-membership function. $s_{[k,l]}$ is equivalent to uncertain linguistic variable $[s_k, s_l]$, and we can express the IVLIFN as $\hat{\gamma}=(s_{[k,l]},s_{[p,q]})$.

In addition, for convenience, we use $\Gamma_{[0,\tau]}$ to express the set of all IVLIFNs.

Definition 12 Let $\hat{\gamma}=(s_{[k,l]},s_{[p,q]})$, $\hat{\gamma}_1=(s_{[k_1,l_1]},s_{[p_1,q_1]})$ and $\hat{\gamma}_2=(s_{[k_2,l_2]},s_{[p_2,q_2]})\in\Gamma_{[0,\tau]}$ be three IVLIFNs, and $\lambda > 0$, then

$$(1)\quad \hat{\gamma}_1\oplus\hat{\gamma}_2=\left(s_{\left[k_1+k_2-\frac{k_1k_2}{\tau},l_1+l_2-\frac{l_1l_2}{\tau}\right]},s_{\left[\frac{p_1p_2}{\tau},\frac{q_1q_2}{\tau}\right]}\right) \quad (19)$$

$$(2)\quad \hat{\gamma}_1\otimes\hat{\gamma}_2=\left(s_{\left[\frac{k_1k_2}{\tau},\frac{l_1l_2}{\tau}\right]},s_{\left[p_1+p_2-\frac{p_1p_2}{\tau},q_1+q_2-\frac{q_1q_2}{\tau}\right]}\right) \quad (20)$$

$$(3)\quad \lambda\hat{\gamma}=\left(s_{\left[\tau-\tau\left(1-\frac{k}{\tau}\right)^{\lambda},\tau-\tau\left(1-\frac{l}{\tau}\right)^{\lambda}\right]},s_{\left[\tau\left(\frac{p}{\tau}\right)^{\lambda},\tau\left(\frac{q}{\tau}\right)^{\lambda}\right]}\right) \quad (21)$$

$$(4)\quad \hat{\gamma}^{\lambda}=\left(s_{\left[\tau\left(\frac{k}{\tau}\right)^{\lambda},\tau\left(\frac{l}{\tau}\right)^{\lambda}\right]},s_{\left[\tau-\tau\left(1-\frac{p}{\tau}\right)^{\lambda},\tau-\tau\left(1-\frac{q}{\tau}\right)^{\lambda}\right]}\right). \quad (22)$$

Example 1 Let $\hat{\gamma}_1=(s_{[3,5]},s_{[1,3]})$, $\hat{\gamma}_2=(s_{[4,5]},s_{[2,3]})\in\Gamma_{[0,8]}$, $\lambda=0.5$, then according to definition12, we can get

$$\hat{\gamma}_1\oplus\hat{\gamma}_2=\left(s_{\left[3+4-\frac{3*4}{8},5+5-\frac{5*5}{8}\right]},s_{\left[\frac{1*2}{8},\frac{3*3}{8}\right]}\right)=\left(s_{[5.5,6.875]},s_{[0.25,1.125]}\right)$$

$$\hat{\gamma}_1\otimes\hat{\gamma}_2=\left(s_{\left[\frac{3*4}{8},\frac{5*5}{8}\right]},s_{\left[1+2-\frac{1*2}{8},3+3-\frac{3*3}{8}\right]}\right)=\left(s_{[1.5,3.125]},s_{[2.75,4.875]}\right)$$

$$\lambda\hat{\gamma}_1=\left(s_{\left[8-8*\left(1-\frac{3}{8}\right)^{0.5},8-8*\left(1-\frac{5}{8}\right)^{0.5}\right]},s_{\left[8*\left(\frac{1}{8}\right)^{0.5},8*\left(\frac{3}{8}\right)^{0.5}\right]}\right)$$
$$=\left(s_{[1.675,3.101]},s_{[2.828,4.899]}\right)$$

$$\hat{\gamma}_1^{\lambda}=\left(s_{\left[8*\left(\frac{3}{8}\right)^{0.5},8*\left(\frac{5}{8}\right)^{0.5}\right]},s_{\left[8-8*\left(1-\frac{1}{8}\right)^{0.5},8-8*\left(1-\frac{3}{8}\right)^{0.5}\right]}\right)$$
$$=\left(s_{[4.899,6.325]},s_{[0.517,1.675]}\right).$$

Next, we give some operational properties for IVLIFNs as follows:

Theorem 1 Let $\hat{\gamma}=(s_{[k,l]},s_{[p,q]})$, $\hat{\gamma}_1=(s_{[k_1,l_1]},s_{[p_1,q_1]})$ and $\hat{\gamma}_2=(s_{[k_2,l_2]},s_{[p_2,q_2]})\in\Gamma_{[0,\tau]}$ be three IVLIFNs, and λ, λ_1, $\lambda_2 > 0$, then

(1) $\hat{\gamma}_1\oplus\hat{\gamma}_2=\hat{\gamma}_2\oplus\hat{\gamma}_1$ (23)

(2) $\hat{\gamma}_1\otimes\hat{\gamma}_2=\hat{\gamma}_2\otimes\hat{\gamma}_1$ (24)

(3) $\hat{\gamma}\oplus(\hat{\gamma}_1\oplus\hat{\gamma}_2)=(\hat{\gamma}\oplus\hat{\gamma}_1)\oplus\hat{\gamma}_2$ (25)

(4) $\hat{\gamma}\otimes(\hat{\gamma}_1\otimes\hat{\gamma}_2)=(\hat{\gamma}\otimes\hat{\gamma}_1)\otimes\hat{\gamma}_2$ (26)

(5) $\lambda(\hat{\gamma}_1\oplus\hat{\gamma}_2)=\lambda\hat{\gamma}_1\oplus\lambda\hat{\gamma}_2$ (27)

(6) $(\hat{\gamma}_1\otimes\hat{\gamma}_2)^{\lambda}=\hat{\gamma}_1^{\lambda}\otimes\hat{\gamma}_2^{\lambda}$ (28)

(7) $\lambda_1\hat{\gamma}\oplus\lambda_2\hat{\gamma}=(\lambda_1+\lambda_2)\hat{\gamma}$ (29)

(8) $\hat{\gamma}^{\lambda_1}\otimes\hat{\gamma}^{\lambda_2}=\hat{\gamma}^{(\lambda_1+\lambda_2)}$ (30)

(9) $\lambda_1\left(\lambda_2\hat{\gamma}\right)=\lambda_1\lambda_2\hat{\gamma}$ (31)

(10) $\left(\hat{\gamma}^{\lambda_1}\right)^{\lambda_2}=\hat{\gamma}^{\lambda_1\lambda_2}.$ (32)

Proof.

1. For (23), according to definition 12, we have

$$\hat{\gamma}_1\oplus\hat{\gamma}_2=\left(s_{[k_1,l_1]},s_{[p_1,q_1]}\right)\oplus\left(s_{[k_2,l_2]},s_{[p_2,q_2]}\right)=\left(s_{\left[k_1+k_2-\frac{k_1k_2}{\tau},l_1+l_2-\frac{l_1l_2}{\tau}\right]},s_{\left[\frac{p_1p_2}{\tau},\frac{q_1q_2}{\tau}\right]}\right)=\left(s_{\left[k_2+k_1-\frac{k_2k_1}{\tau},l_2+l_1-\frac{l_2l_1}{\tau}\right]},s_{\left[\frac{p_2p_1}{\tau},\frac{q_2q_1}{\tau}\right]}\right)$$
$$=\left(s_{[k_2,l_2]},s_{[p_2,q_2]}\right)\oplus\left(s_{[k_1,l_1]},s_{[p_1,q_1]}\right)=\hat{\gamma}_2\oplus\hat{\gamma}_1.$$

2. For (24), there are the same steps of proof as (23); it is omitted here.
3. For the left hand of (25), according to definition 12, we have

$$\hat{\gamma}\oplus\left(\hat{\gamma}_1\oplus\hat{\gamma}_2\right)=\left(s_{[k,l]},s_{[p,q]}\right)\oplus\left[\left(s_{[k_1,l_1]},s_{[p_1,q_1]}\right)\oplus\left(s_{[k_2,l_2]},s_{[p_2,q_2]}\right)\right]$$
$$=\left(s_{[k,l]},s_{[p,q]}\right)\oplus\left(s_{\left[k_1+k_2-\frac{k_1k_2}{\tau},l_1+l_2-\frac{l_1l_2}{\tau}\right]},s_{\left[\frac{p_1p_2}{\tau},\frac{q_1q_2}{\tau}\right]}\right)$$
$$=\left(s_{\left[k+k_1+k_2-\frac{k_1k_2}{\tau}-\frac{kk_1}{\tau}-\frac{kk_2}{\tau}+\frac{kk_1k_2}{\tau^2},l+l_1+l_2-\frac{l_1l_2}{\tau}-\frac{ll_1}{\tau}-\frac{ll_2}{\tau}+\frac{ll_1l_2}{\tau^2}\right]},s_{\left[\frac{pp_1p_2}{\tau^2},\frac{qq_1q_2}{\tau^2}\right]}\right)$$

and for the right hand of (25), we have

$$\left(\hat{\gamma}\oplus\hat{\gamma}_1\right)\oplus\hat{\gamma}_2=\left[\left(s_{[k,l]},s_{[p,q]}\right)\oplus\left(s_{[k_1,l_1]},s_{[p_1,q_1]}\right)\right]\oplus\left(s_{[k_2,l_2]},s_{[p_2,q_2]}\right)$$
$$=\left(s_{\left[k+k_1-\frac{kk_1}{\tau},l+l_1-\frac{ll_1}{\tau}\right]},s_{\left[\frac{pp_1}{\tau},\frac{qq_1}{\tau}\right]}\right)\oplus\left(s_{[k_2,l_2]},s_{[p_2,q_2]}\right)$$
$$=\left(s_{\left[k+k_1+k_2-\frac{k_1k_2}{\tau}-\frac{kk_1}{\tau}-\frac{kk_2}{\tau}+\frac{kk_1k_2}{\tau^2},l+l_1+l_2-\frac{l_1l_2}{\tau}-\frac{ll_1}{\tau}-\frac{ll_2}{\tau}+\frac{ll_1l_2}{\tau^2}\right]},s_{\left[\frac{pp_1p_2}{\tau^2},\frac{qq_1q_2}{\tau^2}\right]}\right)$$

Therefore, we have $\hat{\gamma}\oplus(\hat{\gamma}_1\oplus\hat{\gamma}_2)=(\hat{\gamma}\oplus\hat{\gamma}_1)\oplus\hat{\gamma}_2$.

4. For (26), there are the same steps of proof as (25), and it is omitted here.
5. For the left hand of (27), according to definition 12, we have

$$\lambda\left(\hat{\gamma}_1\oplus\hat{\gamma}_2\right)=\lambda\left[\left(s_{[k1,l1]},s_{[p1,q1]}\right)\oplus\left(s_{[k2,l2]},s_{[p2,q2]}\right)\right]$$
$$=\lambda\left(s_{\left[k_1+k_2-\frac{k_1k_2}{\tau},l_1+l_2-\frac{l_1l_2}{\tau}\right]},s_{\left[\frac{p_1p_2}{\tau},\frac{q_1q_2}{\tau}\right]}\right)=\left(s_{\left[\tau-\tau\left(1-\frac{k_1+k_2-\frac{k_1k_2}{\tau}}{\tau}\right)^{\lambda},\tau-\tau\left(1-\frac{l_1+l_2-\frac{l_1l_2}{\tau}}{\tau}\right)^{\lambda}\right]},s_{\left[\tau\left(\frac{\frac{p_1p_2}{\tau}}{\tau}\right)^{\lambda},t\left(\frac{\frac{q_1q_2}{\tau}}{\tau}\right)^{\lambda}\right]}\right)$$
$$=\left(s_{\left[\tau-\tau\left[\left(1-\frac{k_1}{\tau}\right)\left(1-\frac{k_2}{\tau}\right)\right]^{\lambda},\tau-\tau\left[\left(1-\frac{l_1}{\tau}\right)\left(1-\frac{l_2}{\tau}\right)\right]^{\lambda}\right]},s_{\left[\tau\left(\frac{p_1p_2}{\tau^2}\right)^{\lambda},\tau\left(\frac{q_1q_2}{\tau^2}\right)^{\lambda}\right]}\right),$$

and for the right hand, we have

$$\lambda\hat{\gamma}_1\oplus\lambda\hat{\gamma}_2=\lambda\left(s_{[k_1,l_1]},s_{[p_1,q_1]}\right)\oplus\lambda\left(s_{[k_2,l_2]},s_{[p_2,q_2]}\right)$$
$$=\left(s_{\left[\tau-\tau\left(1-\frac{k_1}{\tau}\right)^{\lambda},\tau-\tau\left(1-\frac{l_1}{\tau}\right)^{\lambda}\right]},s_{\left[\tau\left(\frac{p_1}{\tau}\right)^{\lambda},\tau\left(\frac{q_1}{\tau}\right)^{\lambda}\right]}\right)\oplus\left(s_{\left[\tau-\tau\left(1-\frac{k_2}{\tau}\right)^{\lambda},\tau-\tau\left(1-\frac{l_2}{\tau}\right)^{\lambda}\right]},s_{\left[\tau\left(\frac{p_2}{\tau}\right)^{\lambda},\tau\left(\frac{q_2}{\tau}\right)^{\lambda}\right]}\right)$$
$$=\left(s_{\left[\tau-\tau\left[\left(1-\frac{k_1}{\tau}\right)\left(1-\frac{k_2}{\tau}\right)\right]^{\lambda},\tau-\tau\left[\left(1-\frac{l_1}{\tau}\right)\left(1-\frac{l_2}{\tau}\right)\right]^{\lambda}\right]},s_{\left[\tau\left(\frac{p_1p_2}{\tau^2}\right)^{\lambda},t\left(\frac{q_1q_2}{\tau^2}\right)^{\lambda}\right]}\right).$$

Therefore, we have $\lambda(\hat{\gamma}_1 \oplus \hat{\gamma}_2) = \lambda\hat{\gamma}_1 \oplus \lambda\hat{\gamma}_2$.

6. For (28), there are the same steps of proof as (27); it is omitted here.

7. For (29), according to definition 12, we have

$$\lambda_1\hat{\gamma} \oplus \lambda_2\hat{\gamma} = \lambda_1\left(s_{[k,l]}, s_{[p,q]}\right) \oplus \lambda_2\left(s_{[k,l]}, s_{[p,q]}\right)$$

$$= \left(s_{\left[\tau-\tau\left(1-\frac{k}{\tau}\right)^{\lambda_1}, t-t\left(1-\frac{l}{\tau}\right)^{\lambda_1}\right]}, s_{\left[\tau\left(\frac{p}{\tau}\right)^{\lambda_1}, \tau\left(\frac{q}{\tau}\right)^{\lambda_1}\right]}\right) \oplus \left(s_{\left[\tau-\tau\left(1-\frac{k}{\tau}\right)^{\lambda_2}, \tau-\tau\left(1-\frac{l}{\tau}\right)^{\lambda_2}\right]}, s_{\left[\tau\left(\frac{p}{\tau}\right)^{\lambda_2}, \tau\left(\frac{q}{\tau}\right)^{\lambda_2}\right]}\right)$$

$$= \left(s_{\left[\left(\tau-\tau\left(1-\frac{k}{\tau}\right)^{\lambda_1}\right)+\left(\tau-\tau\left(1-\frac{k}{\tau}\right)^{\lambda_2}\right)-\frac{\left(\tau-\tau\left(1-\frac{k}{\tau}\right)^{\lambda_1}\right)*\left(\tau-\tau\left(1-\frac{k}{\tau}\right)^{\lambda_2}\right)}{\tau}, \left(\tau-\tau\left(1-\frac{l}{\tau}\right)^{\lambda_1}\right)+\left(\tau-\tau\left(1-\frac{l}{\tau}\right)^{\lambda_2}\right)-\frac{\left(\tau-\tau\left(1-\frac{l}{\tau}\right)^{\lambda_1}\right)*\left(\tau-\tau\left(1-\frac{l}{\tau}\right)^{\lambda_2}\right)}{\tau}\right]}, s_{\left[\frac{\tau\left(\frac{p}{\tau}\right)^{\lambda_1}\tau\left(\frac{p}{\tau}\right)^{\lambda_2}}{\tau}, \frac{\tau\left(\frac{q}{\tau}\right)^{\lambda_1}\tau\left(\frac{q}{\tau}\right)^{\lambda_2}}{\tau}\right]}\right)$$

$$= \left(s_{\left[\tau-\tau\left(1-\frac{k}{\tau}\right)^{\lambda_1+\lambda_2}, \tau-\tau\left(1-\frac{l}{\tau}\right)^{\lambda_1+\lambda_2}\right]}, s_{\left[\tau\left(\frac{p}{\tau}\right)^{\lambda_1+\lambda_2}, \tau\left(\frac{q}{\tau}\right)^{\lambda_1+\lambda_2}\right]}\right) = (\lambda_1+\lambda_2)\left(s_{[k,l]}, s_{[p,q]}\right) = (\lambda_1+\lambda_2)\hat{\gamma}$$

8. For (30), there are the same steps of proof as (29); it is omitted here.

9. For (31), according to definition 12, we have

$$\lambda_1\left(\lambda_2\hat{\gamma}\right) = \lambda_1\left[\lambda_2\left(s_{[k,l]}, s_{[p,q]}\right)\right] = \lambda_1\left(s_{\left[\tau-\tau\left(1-\frac{k}{\tau}\right)^{\lambda_2}, \tau-\tau\left(1-\frac{l}{\tau}\right)^{\lambda_2}\right]}, s_{\left[\tau\left(\frac{p}{\tau}\right)^{\lambda_2}, \tau\left(\frac{q}{\tau}\right)^{\lambda_2}\right]}\right)$$

$$= \left(s_{\left[\tau-\tau\left(\frac{\tau-\tau\left(1-\frac{k}{\tau}\right)^{\lambda_2}}{\tau}\right)^{\lambda_1}, \tau-\tau\left(\frac{\tau-\tau\left(1-\frac{l}{\tau}\right)^{\lambda_2}}{\tau}\right)^{\lambda_1}\right]}, s_{\left[\tau\left(\frac{\tau\left(\frac{p}{\tau}\right)^{\lambda_2}}{\tau}\right)^{\lambda_1}, \tau\left(\frac{\tau\left(\frac{q}{\tau}\right)^{\lambda_2}}{\tau}\right)^{\lambda_1}\right]}\right) = \left(s_{\left[\tau-\tau\left(1-\frac{k}{\tau}\right)^{\lambda_1+\lambda_2}, \tau-\tau\left(1-\frac{l}{\tau}\right)^{\lambda_1+\lambda_2}\right]}, s_{\left[\tau\left(\frac{p}{\tau}\right)^{\lambda_1+\lambda_2}, \tau\left(\frac{q}{\tau}\right)^{\lambda_1+\lambda_2}\right]}\right)$$

$$= \lambda_1\lambda_2\left(s_{[k,l]}, s_{[p,q]}\right) = \lambda_1\lambda_2\hat{\gamma}.$$

10. For (32), there are the same steps of proof as (30); it is omitted here.

Next, we will give a comparison method for two IVLIFNs as follows.

Definition 13 Let $\hat{\gamma} = \left(s_{[k,l]}, s_{[p,q]}\right) \in \Gamma_{[0,\tau]}$ be an IVLIFN, then we have the score value $Ivls(\hat{\gamma})$ and accuracy value $Ivlh(\hat{\gamma})$ of $\hat{\gamma}$ as follows:

$$Ivls\left(\hat{\gamma}\right) = \frac{k+l-p-q}{2}; \tag{33}$$

$$Ivlh\left(\hat{\gamma}\right) = \frac{k+l+p+q}{2}. \tag{34}$$

Definition 14 Let $\hat{\gamma}_1 = \left(s_{[k_1,l_1]}, s_{[p_1,q_1]}\right)$ and $\hat{\gamma}_2 = \left(s_{[k_2,l_2]}, s_{[p_2,q_2]}\right) \in \Gamma_{[0,\tau]}$ be any two IVLIFNs, we have

1. If $Ivls(\hat{\gamma}_1) > Ivls(\hat{\gamma}_2)$, then $\hat{\gamma}_1 \succ \hat{\gamma}_2$;
2. If $Ivls(\hat{\gamma}_1) = Ivls(\hat{\gamma}_2)$,

If $Ivlh(\hat{\gamma}_1) > Ivlh(\hat{\gamma}_2)$, then $\hat{\gamma}_1 \succ \hat{\gamma}_2$;
If $Ivlh(\hat{\gamma}_1) = Ivlh(\hat{\gamma}_2)$, then $\hat{\gamma}_1 = \hat{\gamma}_2$.

Example 2 Let $\hat{\gamma}_1 = \left(s_{[3,5]}, s_{[1,3]}\right), \hat{\gamma}_2 = \left(s_{[4,5]}, s_{[2,3]}\right), \hat{\gamma}_3 = \left(s_{[5,6]}, s_{[1,2]}\right) \in S_{[0,8]}$, we can get that

$Ivls(\hat{\gamma}_1) = 2, Ivls(\hat{\gamma}_2) = 2, Ivls(\hat{\gamma}_3) = 4;$ $Ivlh(\hat{\gamma}_1) = 6, Ivlh(\hat{\gamma}_2) = 7, Ivls(\hat{\gamma}_3) = 7.$

Then, we can conclude that $\hat{\gamma}_1 \prec \hat{\gamma}_2 \prec \hat{\gamma}_3$.

Some Maclaurin Symmetric Mean Operator Aggregation Operators of IVLIFNs

The MSM has the good property in considering the interrelationships among any number of aggregated arguments. It is more flexible and can be simplified to the special cases, such as arithmetic aggregation operator when $x = 1$, BM operator when $x = 2$, and geometric aggregation operator when $x = n$. However, it can only deal with the crisp number, and not IVLIFNs. In this part, we develop two new MSM operators for IVLIFNs, such as the IVLIFMSM operator and the WIVLIFMSM operator, and prove some effective properties. In addition, we explored some special cases of these operators according to adjusting the parameter x.

The Interval-Valued Linguistic Intuitionistic Fuzzy Maclaurin Symmetric Mean Operator

Definition 15 Let $\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n \in \Gamma_{(0,\tau)}$ be a set of IVLIFNs, and then the IVLIFMSM operator can be defined as follows:

$$IVLIFMSM^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right) = \left(\frac{\underset{1 \le i_1 \prec ... \prec i_x \le n}{\oplus} \ \overset{x}{\underset{j=1}{\otimes}} \hat{\gamma}_{i_j}}{C_n^x}\right)^{\frac{1}{x}} \tag{35}$$

where $(i_1, i_2, ..., i_n)$ includes all the x-tuple combination of $(1, 2, ..., n)$, and C_n^k is the binomial coefficient.

Theorem 2 Let $\hat{\gamma}_i = \left(s_{[k_i,l_i]}, s_{[p_i,q_i]}\right)$ $(i = 1, 2, 3, ..., n)$ be a set of IVLIFNs; the aggregation result from definition15 is an IVLIFN, and has

$$IVLIFMSM^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right) = \left(s_{\left[\tau\left(1-\left(\prod_{1 \le i_1 \prec ... \prec i_x \le n}\left(1-\frac{\prod_{j=1}^{x} k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}, \tau\left(1-\left(\prod_{1 \le i_1 \prec ... \prec i_x \le n}\left(1-\frac{\prod_{j=1}^{x} l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]}, s_{\left[\tau-\tau\left(1-\left(\prod_{1 \le i_1 \prec ... \prec i_x \le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}, \tau-\tau\left(1-\left(\prod_{1 \le i_1 \prec ... \prec i_x \le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]} \right) \tag{36}$$

Proof.

In order to prove Eq. (36), we give another theorem firstly.

Theorem 3 Let $\hat{\gamma}_i = \left(s_{[k_i,l_i]}, s_{[p_i,q_i]}\right)$ $(i = 1, ..., n)$ be a set of IVLIFNs, then

$$\sum_{i=1}^{n} \hat{\gamma}_i = \left(s_{\left[\tau-\tau\prod_{i=1}^{n}\left(1-\frac{k_i}{\tau}\right), \tau-\tau\prod_{i=1}^{n}\left(1-\frac{l_i}{\tau}\right)\right]}, s_{\left[\tau\prod_{i=1}^{n}\left(\frac{p_i}{\tau}\right), \tau\prod_{i=1}^{n}\left(\frac{q_i}{\tau}\right)\right]} \right) \tag{37}$$

The proof is omitted here.

For theorem 2, we need to prove that (1) Eq. (36) is kept and (2) the result of Eq. (36) is still an IVLIFN.

1. We prove that Eq. (36) is kept.

According to the operational laws of IVLIFNs, we get

$$\overset{x}{\underset{j=1}{\otimes}} \hat{\gamma}_{i_j} = \left(s_{\left[\frac{\prod_{j=1}^{x} k_{i_j}}{\tau^{x-1}}, \frac{\prod_{j=1}^{x} l_{i_j}}{\tau^{x-1}}\right]}, s_{\left[\tau-\tau\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right), \tau-\tau\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right]} \right),$$

and according to Eq. (37), we have

$$\underset{1\le i_1\prec\ldots\prec i_x\le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}}\hat{\gamma}_{i_j}\right)=\left(s_{\left[\tau-\tau\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right),\tau-\tau\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\right]},s_{\left[\tau\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right)\right),\tau\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right]}\right)$$

then

$$\frac{1}{C_n^x}\left(\underset{1\le i_1\prec\ldots\prec i_x\le n}{\oplus}\left(\overset{x}{\underset{j=1}{\otimes}}\hat{\gamma}_{i_j}\right)\right)=\left(s_{\left[\tau-\tau\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}},\tau-\tau\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right]},\right.$$
$$\left.s_{\left[\tau\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}},\tau\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right]}\right)$$

Therefore,

$$IVLIFMSM^{(x)}\left(\hat{\gamma}_1,\hat{\gamma}_2,\ldots,\hat{\gamma}_n\right)=\left(s_{\left[\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}},\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]},\right.$$
$$\left.s_{\left[\tau-\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}},\tau-\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]}\right)$$

---

2. We prove that Eq. (36) is still an IVLIFN.

Let $IVLIFMSM^{(k)}(\hat{\gamma}_1,\hat{\gamma}_2,\ldots,\hat{\gamma}_n)=\left(s_{[k,l]},s_{[p,q]}\right)$, then we need to prove (i) $[k,l]\subset[0,t]$, $[p,q]\subset[0,t]$ and (ii) $l+q<t$.

(i) We prove $[k,l]\subset[0,t]$, $[p,q]\subset[0,t]$.

Since $\left[k_{i_j},l_{i_j}\right]\subset[0,t]$, we can get

$$\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\in[0,1],\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\in[0,1],$$

and because $k_{i_j}\le l_{i_j}$, then $\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}$

$$\ge\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}},$$

Further,

$$1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\in[0,1],\ 1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\in[0,1]$$

and

$$1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\le 1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}.$$

So, we have

$$\left[\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}},\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]\subset[0,1].$$

Then,

$$[k,l]=\left[\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}},\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]\subset[0,\tau].$$

Similarly, we can get

$$[p,q]=\left[\tau-\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}},\tau-\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]\subset[0,\tau].$$

(ii) We prove $l+q<t$.
Since $0\le l_{i_j}+q_{i_j}\le\tau$, we can get

$$\begin{aligned}
l+q&=\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}+\tau-\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\\
&\le\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x}\left(\tau-q_{i_j}\right)}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}+\tau-\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\\
&=\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}+\tau-\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\\
&=\tau
\end{aligned}$$

Based on the proofs of (i) and (ii), we get Eq. (36) is still an IVLIFN.

So, according to (1) and (2), theorem 2 is kept.

Next, we explore some properties of IVLIFMSM operator.

Property 1 (Idempotency). Let $\hat{\gamma}_i = \left(s_{[k_i,l_i]}, s_{[p_i,q_i]}\right)$ $(i = 1, ..., n)$ be a set of IVLIFNs, if $\hat{\gamma}_1 = \hat{\gamma}_2 = ... = \hat{\gamma}_n = \hat{\gamma} = \left(s_{[k,l]}, s_{[p,q]}\right)$, then

$$IVLIFMSM^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right) = \hat{\gamma} \tag{38}$$

Proof. Since $\hat{\gamma} = \left(s_{[k,l]}, s_{[p,q]}\right)$, based on theorem 2, we have

$$\begin{aligned}
&IVLIFMSM^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right)\\
&= \left(s_{\left[\tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x} k}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}, \tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{k} l}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]}, s_{\left[\tau-\tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}, \tau-\tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]} \right)\\
&= \left(s_{\left[\tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\left(\frac{k}{\tau}\right)^x\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}, t\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\left(\frac{l}{\tau}\right)^x\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]}, s_{\left[\tau-\tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\left(1-\frac{p}{\tau}\right)^x\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}, \tau-\tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\left(1-\frac{q}{\tau}\right)^x\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]} \right)\\
&= \left(s_{\left[\tau\left(1-\left(1-\left(\frac{k}{\tau}\right)^x\right)\right)^{\frac{1}{x}}, \tau\left(1-\left(1-\left(\frac{l}{\tau}\right)^x\right)\right)^{\frac{1}{x}}\right]}, s_{\left[\tau-\tau\left(1-\left(1-\left(1-\frac{p}{\tau}\right)^x\right)\right)^{\frac{1}{x}}, \tau-\tau\left(1-\left(1-\left(1-\frac{q}{\tau}\right)^x\right)\right)^{\frac{1}{x}}\right]} \right)\\
&= \left(s_{[k,l]}, s_{[p,q]}\right) = \hat{\gamma}.
\end{aligned}$$

Property 2 (Monotonicity). Suppose $\hat{\gamma}_i = \left(s_{[k_i,l_i]}, s_{[p_i,q_i]}\right)$ and $\hat{\gamma}_i{'}' = \left(s_{\left[k_i'',l_i''\right]}, s_{\left[p_i'',q_i''\right]}\right)$ are two sets of IVLIFNs. If $s_{k_i''} \ge s_{k_i}, s_{l_i''} \ge s_{l_i}$ and $s_{p_i''} \le s_{p_i}, s_{q_i''} \le s_{q_i}$ for all i, then

$$IVLIFMSM^{(x)}\left(\hat{\gamma}_1'', \hat{\gamma}_2'', ..., \hat{\gamma}_n''\right) \ge IVLIFMSM^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right) \tag{39}$$

Proof.

(1) For the membership function of $IVLIFMSM^{(x)}\left(\hat{\gamma}_1'', \hat{\gamma}_2'', ..., \hat{\gamma}_n''\right)$ and $IVLIFMSM^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right)$,

Since $k \ge 1, s_{k_i''} \ge s_{k_i}, s_{l_i''} \ge s_{l_i}$, then $k_i{'}' \ge k_i, l_i'' \ge l_i$, and have

$$\frac{\prod_{j=1}^{x} k_{i_j}''}{\tau^x} \ge \frac{\prod_{j=1}^{x} k_{i_j}}{\tau^x} \text{ and } 1-\frac{\prod_{j=1}^{x} k_{i_j}''}{\tau^x} \le 1-\frac{\prod_{j=1}^{x} k_{i_j}}{\tau^x},$$

then $\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x} k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}} \le \left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x} k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}$,

further, $1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x} k_{i_j}''}{\tau^x}\right)\right)^{\frac{1}{C_n^x}} \ge 1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x} k_{i_j}''}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}$

so we have $\tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x} k_{i_j}''}{\tau^x}\right)\right)^{\frac{\prod_{j=1}^{x} k_{i_j}''}{\tau^x}}\right) \ge \tau$

$$\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x} k_{i_j}''}{\tau^x}\right)\right)^{\frac{\prod_{j=1}^{x} k_{i_j}''}{\tau^x}}\right).$$

Similarly, we get

$$t\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x} l_{i_j}''}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)$$

$$\ge t\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\frac{\prod_{j=1}^{x} l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right).$$

(2) For the non-membership function of $IVLIFMSM^{(x)}(\hat{\gamma}''_1, \hat{\gamma}''_2, ..., \hat{\gamma}''_n)$ and $IVLIFMSM^{(x)}(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n)$, since $s_{p''_i} \le s_{p_i}, s_{q''_i} \le s_{q_i}$, then we have $p''_i \le p_i, q''_i \le q_i$ and

$$1-\frac{p''_{i_j}}{\tau} \ge 1-\frac{p_{i_j}}{\tau} \Rightarrow \prod_{j=1}^{x}\left(1-\frac{p''_{i_j}}{\tau}\right) \ge \prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right) \Rightarrow 1-\prod_{j=1}^{x}\left(1-\frac{p''_{i_j}}{\tau}\right) \le 1-\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right)$$

$$\Rightarrow\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p''_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}} \le \left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}$$

$$\Rightarrow t\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p''_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}} \ge t\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}},$$

$$\Rightarrow \tau-\tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p''_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}} \le \tau-\tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}};$$

Similarly, we get

$$\tau-\tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}} \le \tau-\tau\left(1-\left(\prod_{1\le i_1\prec...\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}$$

(3) According to definitions 13 and 1, we can get

$$IVLIFMSM^{(x)}\left(\hat{\gamma}''_1, \hat{\gamma}''_2, ..., \hat{\gamma}''_n\right) \ge IVLIFMSM^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right).$$

Property 3 (Boundedness). Suppose $\hat{\gamma}_i = \left(s_{[k_i,l_i]}, s_{[p_i,q_i]}\right)$ is a set of IVLIFNs, and $\hat{\gamma}^- = \left(s_{[\min k_i, \min l_i]}, s_{[\max p_i, \max q_i]}\right), \hat{\gamma}^+ = \left(s_{[\max k_i, \max l_i]}, s_{[\min p_i, \min q_i]}\right)$, then

$$\hat{\gamma}^- \le IVLIFMSM^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right) \le \hat{\gamma}^+ \tag{40}$$

Proof.

Based on properties 1 and 2, we have

$$IVLIFMSM^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right) \ge IVLIFMSM^{(x)}\left(\hat{\gamma}^-, \hat{\gamma}^-, ..., \hat{\gamma}^-\right) = \hat{\gamma}^-,$$

And $IVLIFMSM^{(x)}(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n) \le IVLIFMSM^{(x)}(\hat{\gamma}^+, \hat{\gamma}^+, ..., \hat{\gamma}^+) = \hat{\gamma}^+$, then we have $\hat{\gamma}^- \le IVLIFMSM^{(x)}(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n) \le \hat{\gamma}^+$.

In the following, in order to discuss the monotonicity of the IVLIFMSM operator for parameter x, we quote two lemmas.

Lemma 1 (Maclaurin inequality) [32]. Let $\psi_i (i = 1\ to\ n)$ be a group of numbers which are non-negative real numbers, we have

$$MSM^{(1)}(\psi_1, \psi_2, ..., \psi_n) \ge MSM^{(2)}(\psi_1, \psi_2, ..., \psi_n) \ge ... \ge MSM^{(x)}(\psi_1, \psi_2, ..., \psi_n) \tag{41}$$

Only when $\psi_1 = \psi_2 = ... = \psi_n$, the equation keeps, where $x = 1\ to\ n$.

Lemma 2 [32]. Let $a_i > 0, b_i > 0 (i = 1\ to\ n)$, $\sum_{i=1}^{n} b_i = 1$,

$$\prod_{i=1}^{n} a_i^{b_i} \le \sum_{i=1}^{n} a_i b_i \tag{42}$$

then, only when $a_1 = a_2 = ... = a_n$, the equation keeps.

Theorem 4 For given IVLIFNs $\hat{\gamma}_i \in \Gamma_{[0,\tau]} (i = 1\ to\ n)$, also $x = 1\ to\ n$, the IVLIFMSM is monotonically decreasing when the parameter x is monotonically increased.

Proof. Based on theorem 2,

$$IVLIFMSM^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right) = \left(s_{\left[\tau\left(1-\left(\prod_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}, \tau\left(1-\left(\prod_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]}, \right.$$

$$\left. s_{\left[\tau-\tau\left(1-\left(\prod_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}, \tau-\tau\left(1-\left(\prod_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]} \right).$$

Let

$$f(x) = \tau\left(1-\left(\prod_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}} + \tau\left(1-\left(\prod_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}$$

and

$$g(x) = \tau-\tau\left(1-\left(\prod_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\prod_{j=1}^{x}\left(1-\frac{p_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}} + \tau-\tau\left(1-\left(\prod_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}$$

Now, we prove that when the parameter x is increased, the function $f(x)$ is decreased.

Based on the Lemma 1 and Lemma 2, we have

$$f(x) = \tau\left(1-\left(\prod_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}} + \tau\left(1-\left(\prod_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}$$

$$\geq t\left(1-\sum_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}\right)\frac{1}{C_n^x}\right)^{\frac{1}{x}} + \tau\left(1-\sum_{1\leq i_1\prec...\prec i_x\leq n}\left(1-\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}\right)\frac{1}{C_n^x}\right)^{\frac{1}{x}}$$

$$= \tau\left(1-\left(1-\sum_{1\leq i_1\prec...\prec i_x\leq n}\left(\frac{\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}}{C_n^x}\right)\right)\right)^{\frac{1}{x}} + \tau\left(1-\left(1-\sum_{1\leq i_1\prec...\prec i_x\leq n}\left(\frac{\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}}{C_n^x}\right)\right)\right)^{\frac{1}{x}}$$

$$= \tau\sum_{1\leq i_1\prec...\prec i_x\leq n}\left(\frac{\frac{\prod_{j=1}^{x}k_{i_j}}{\tau^x}}{C_n^x}\right)^{\frac{1}{x}} + \tau\sum_{1\leq i_1\prec...\prec i_x\leq n}\left(\frac{\frac{\prod_{j=1}^{x}l_{i_j}}{\tau^x}}{C_n^x}\right)^{\frac{1}{x}}$$

Then, we give the following proof by the reduction to absurdity. Suppose when the parameter x is increasing, the $f(x)$ is following increasing, then $f(x) > f(x-1) > \ldots > f(1)$,also since

$$f(1) \geq \tau \sum_{1 \leq i_1 \leq n} \left(\frac{\frac{\prod_{j=1}^{1} k_{i_j}}{\tau^1}}{C_n^1} \right)^{\frac{1}{1}} + \tau \sum_{1 \leq i_1 \leq n} \left(\frac{\frac{\prod_{j=1}^{1} l_{i_j}}{\tau^1}}{C_n^1} \right)^{\frac{1}{1}} = \frac{\sum_{i=1}^{n} k_i}{n} + \frac{\sum_{i=1}^{n} l_i}{n}$$

then it follows that

$$f(x) > f(1) \geq t \sum_{1 \leq i_1 \leq n} \left(\frac{\frac{\prod_{j=1}^{1} k_{i_j}}{\tau^1}}{C_n^1} \right)^{\frac{1}{1}} = \frac{\sum_{i=1}^{n} k_i + \sum_{i=1}^{n} l_i}{x}$$

$$\Rightarrow \left(\prod_{i=1}^{n} k_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} l_i\right)^{\frac{1}{n}} > \frac{\prod_{i=1}^{n} k_i}{n} + \frac{\prod_{i=1}^{n} l_i}{n}$$

But due to the inequality [41, 42], we have $\left(\prod_{i=1}^{n} k_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} l_i\right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} k_i + \sum_{i=1}^{n} l_i}{n}$.

Obviously, it is false that the $f(x)$ is monotonically increasing. So, the $f(x)$ is decreasing with the parameter x increasing monotonically.

Similarly, we can also prove $g(x)$ is increasing with the parameter x increased.

By using the score value of IVLIFNs, we can know $Ivls(x) = \frac{1}{2}(f(x) - g(x))$ for any $x \in [1, n]$ and$x$is integer number, then we can obtain

$$Ivls(x+1) - Ivls(x) = \frac{1}{2}(f(x+1) - g(x+1)) - (f(x) - g(x))$$

$$= \frac{1}{2}(f(x+1) - f(x)) + (-g(x+1) + g(x))$$

With the parameter x increasing, $f(x)$ is decreasing, so $f(x+1) < f(x)$; with the parameter x increasing,$g(x)$ is increasing, so we can imply $g(x) < g(x+1)$.

Based on the results above, we have $Ivls(x+1) - Ivls(x) = \frac{1}{2}(f(x+1) - f(x)) + (-g(x+1) + g(x)) < 0$

i.e., $Ivls(x+1) < Ivls(x)$, for all x.

Thus, the IVLIFMSM is decreasing with respect to the parameter x.

Next, we present some specific cases of the IVLIFMSM operator with different parameter x.

Case 1. When $x = 1$, the IVLIFMSM operator will reduce to arithmetic average operator of IVLIFNs shown as follows [47]:

$$IVLIFMSM^{(1)}\left(\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_n\right) = \frac{1}{n} \sum_{i=1}^{n} \hat{\gamma}_i \tag{43}$$

$$= \left(s_{\left[\tau - \tau\left(\prod_{i=1}^{n}\left(1 - \frac{k_i}{\tau}\right)\right)^{\frac{1}{n}}, \tau - \tau\left(\prod_{i=1}^{n}\left(1 - \frac{l_i}{\tau}\right)\right)^{\frac{1}{n}}\right]}, s_{\left[\tau\left(\prod_{i=1}^{n}\frac{p_i}{\tau}\right)^{\frac{1}{n}}, \tau\left(\prod_{i=1}^{n}\frac{q_i}{\tau}\right)^{\frac{1}{n}}\right]} \right)$$

Case 2. When $x = 2$, the IVLIFMSM operator will reduce to special BM (when $p = q = 1$) of IVLIFNs shown as follows [7]:

$$IVLIFMSM^{(2)}\left(\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_n\right) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \hat{\gamma}_i \hat{\gamma}_j \right)^{\frac{1}{2}} = \left(\begin{array}{l} s_{\left[\tau\left(1 - \left(\prod_{\substack{i_1 = i_2 = 1 \\ i_1 \neq i_2}}^{n} \frac{k_{i_1} k_{i_2}}{\tau^3}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}, \tau\left(1 - \left(\prod_{\substack{i_1 = i_2 = 1 \\ i_1 \neq i_2}}^{n} \frac{l_{i_1} l_{i_2}}{\tau^3}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right]}, \\ s_{\left[\tau - \tau\left(1 - \left(\prod_{\substack{i_1 = i_2 = 1 \\ i_1 \neq i_2}}^{n} 1 - \left(1 - \frac{p_{i_1}}{\tau}\right)\left(1 - \frac{p_{i_2}}{\tau}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}, \tau - \tau\left(1 - \left(\prod_{\substack{i_1 = i_2 = 1 \\ i_1 \neq i_2}}^{n} 1 - \left(1 - \frac{q_{i_1}}{\tau}\right)\left(1 - \frac{q_{i_2}}{\tau}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right]} \end{array} \right) \tag{44}$$

Case 3. When $x = n$, the IVLIFMSM operator will reduce to geometric average operator of IVLIFNs shown as follows [42]:

$$LIFMSM^{(n)}\left(\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_n\right) = \prod_{i=1}^{n} \hat{\gamma}_i^{\frac{1}{n}} = \left(s_{\left[\tau\left(\frac{\prod_{i=1}^{n} k_i}{\tau^n}\right)^{\frac{1}{n}}, \tau\left(\frac{\prod_{i=1}^{n} l_i}{\tau^n}\right)^{\frac{1}{n}}\right]}, s_{\left[\tau - \tau\left(\prod_{i=1}^{n}\left(1 - \frac{p_i}{\tau}\right)\right)^{\frac{1}{n}}, \tau - \tau\left(\prod_{i=1}^{n}\left(1 - \frac{q_i}{\tau}\right)\right)^{\frac{1}{n}}\right]} \right) \tag{45}$$

Example 3 Let $\hat{\gamma}_1 = \left(s_{[5,7]}, s_{[0,1]}\right), \hat{\gamma}_2 = \left(s_{[5,6]}, s_{[1,2]}\right), \hat{\gamma}_3 = \left(s_{[5,6]}, s_{[0,1]}\right)$ and $\hat{\gamma}_4 = \left(s_{[4,5]}, s_{[1,3]}\right) \in \Gamma_{[0,8]}$ be four IVLIFNs, then we use the proposed IVLIFMSM operator to aggregate them, and get (suppose $x = 2$).

$$\hat{\gamma}_1 \otimes \hat{\gamma}_2 = \left(s_{\left[\frac{5\times5}{8},\frac{7\times6}{8}\right]}, s_{\left[0+1-\frac{0\times1}{8},1+2-\frac{1\times2}{8}\right]}\right) = \left(s_{[3.125,5.25]}, s_{[1,2.75]}\right), \hat{\gamma}_1 \otimes \hat{\gamma}_3 = \left(s_{\left[\frac{5\times5}{8},\frac{7\times6}{8}\right]}, s_{\left[0+0-\frac{0\times0}{8},1+1-\frac{1\times1}{8}\right]}\right) = \left(s_{[3.125,5.25]}, s_{[0,1.875]}\right),$$

$$\hat{\gamma}_1 \otimes \hat{\gamma}_4 = \left(s_{\left[\frac{5\times4}{8},\frac{7\times5}{8}\right]}, s_{\left[0+1-\frac{0\times1}{8},1+3-\frac{1\times3}{8}\right]}\right) = \left(s_{[2.5,4.375]}, s_{[1,3.625]}\right), \hat{\gamma}_2 \otimes \hat{\gamma}_3 = \left(s_{\left[\frac{5\times5}{8},\frac{6\times6}{8}\right]}, s_{\left[1+0-\frac{1\times0}{8},2+1-\frac{2\times1}{8}\right]}\right) = \left(s_{[3.125,4.5]}, s_{[1,2.75]}\right),$$

$$\hat{\gamma}_2 \otimes \hat{\gamma}_4 = \left(s_{\left[\frac{5\times4}{8},\frac{6\times5}{8}\right]}, s_{\left[1+1-\frac{1\times1}{8},2+3-\frac{2\times3}{8}\right]}\right) = \left(s_{[2.5,3.75]}, s_{[1.875,4.25]}\right), \hat{\gamma}_3 \otimes \hat{\gamma}_4 = \left(s_{\left[\frac{5\times4}{8},\frac{6\times5}{8}\right]}, s_{\left[0+1-\frac{0\times1}{8},1+3-\frac{1\times3}{8}\right]}\right) = \left(s_{[2.5,3.75]}, s_{[1,3.625]}\right).$$

Then, by Eq. (36), we have

$$IVLIFMSM^{(2)}\left(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4\right) = \left(\frac{\underset{1 \le i_1 \prec i_2 \le 4}{\oplus} \left(\hat{\gamma}_{i_1} \otimes \hat{\gamma}_{i_2}\right)}{6}\right)^{\frac{1}{2}}$$

$$= \left(s_{[4.751,6.023]}, s_{[0,1.7012]}\right).$$

The Weighted Interval-valued Linguistic Intuitionistic Fuzzy Maclaurin Symmetric Mean Operator

In MADM problems, weight vector of attributes is an important parameter, and it can influence decision-making results. However, the IVLIFMSM does not take the weight vector into account. In order to overcome this limitation, we will define the WIVLIFMSM operator.

Definition 16 Suppose $(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n)$ is a set of IVLIFNs, $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight set of $(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n)$, which satisfies $\omega_i \in [0, 1]$, $i = 1$ *to* n.and $\sum_{i=1}^{n} \omega_i = 1$, then the WIVLIFMSM operator can be presented as follows.

$$WIVLIFMSM_{\omega}^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right) = \left(\frac{\underset{1 \le i_1 \prec \ldots \prec i_x \le n}{\oplus} \overset{x}{\underset{j=1}{\otimes}} n\omega_{i_j} \hat{\gamma}_{i_j}}{C_n^x}\right)^{\frac{1}{x}} \tag{46}$$

Theorem 5 Assume $\hat{\gamma}_i = \left(s_{[k_i,l_i]}, s_{[p_i,q_i]}\right)$ $(i = 1\ to\ n)$ is a group of IVLIFNs, the aggregation result from definition 16 is still an IVLIFN, and we have

$$WIVLIFMSM_{\omega}^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right) = \left(s_{\left[\tau\left(1-\left(\prod_{1 \le i_1 \prec \ldots \prec i_x \le n}\left(1-\prod_{j=1}^{x}\left(1-\left(1-\frac{k_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}, \tau\left(1-\left(\prod_{1 \le i_1 \prec \ldots \prec i_x \le n}\left(1-\prod_{j=1}^{x}\left(1-\left(1-\frac{l_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]}, s_{\left[\tau-\tau\left(1-\left(\prod_{1 \le i_1 \prec \ldots \prec i_x \le n}\left(1-\prod_{j=1}^{x}\left(1-\left(\frac{p_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}, \tau-\tau\left(1-\left(\prod_{1 \le i_1 \prec \ldots \prec i_x \le n}\left(1-\prod_{j=1}^{x}\left(1-\frac{q_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]} \right) \tag{47}$$

For the proof, it is similar to theorem 2; it was omitted here.

Next, we shall deduce some desirable properties of the WIVLIFMSM operator.

Property 4. (Monotonicity) Suppose $\hat{\gamma}_i = \left(s_{[k_i,l_i]}, s_{[p_i,q_i]}\right)$, $\hat{\gamma}_i^{''} = \left(s_{\left[k_i^{''},l_i^{''}\right]}, s_{\left[p_i^{''},q_i^{''}\right]}\right)$ $(i = 1\ to\ n)$ are two groups of IVLIFNs. If $s_{k_i^{''}} \ge s_{k_i}, s_{l_i^{''}} \ge s_{l_i}$ and $s_{p_i^{''}} \le s_{p_i}, s_{q_i^{''}} \le s_{q_i}$ for all i, then

$$WIVLIFMSM_{\omega}^{(x)}\left(\hat{\gamma}_1^{''}, \hat{\gamma}_2^{''}, ..., \hat{\gamma}_n^{''}\right) \ge WIVLIFMSM_{\omega}^{(x)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right) \tag{48}$$

Property 5. (Boundedness) Suppose $\hat{\gamma}_i = \left(s_{[k_i,l_i]}, s_{[p_i,q_i]}\right)$ $(i = 1\ to\ n)$ is a set of n IVLIFNs, and let

$$\hat{\gamma}^{-}=\left(s_{\left[\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}1-\left(1-\frac{\min k_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}},\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}1-\left(1-\frac{\min l_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]},\right.$$
$$\left.s_{\left[\tau-\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\left(\frac{\min p_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}},\tau-\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\left(\frac{\min q_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]}\right)$$

$$\hat{\gamma}^{+}=\left(s_{\left[\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}1-\left(1-\frac{\max k_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}},\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}1-\left(1-\frac{\max l_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]},\right.$$
$$\left.s_{\left[\tau-\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\left(\frac{\max p_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}},\tau-\tau\left(1-\left(\prod_{1\le i_1\prec\ldots\prec i_x\le n}\left(1-\prod_{j=1}^{x}\left(1-\left(\frac{\max q_{i_j}}{\tau}\right)^{n\omega_{i_j}}\right)\right)\right)^{\frac{1}{C_n^x}}\right)^{\frac{1}{x}}\right]}\right)$$

Then,

$$\hat{\gamma}^{-}\le WIVLIFMSM_{\omega}^{(x)}\left(\hat{\gamma}_1,\hat{\gamma}_2,\ldots,\hat{\gamma}_n\right)\le\hat{\gamma}^{+} \tag{49}$$

Theorem 6 For given the set of n IVLIFNs $\hat{\gamma}_i\in\Gamma_{[0,\tau]}(i=1\ to\ n)$, and $x=1$ to n, the WIVLIFMSM is decreasing when the parameter x is increased.

We present some specific cases of the IVLIFMSM operator with different parameter x as follows:

Case 4. When $x=1$, the WIVLIFMSM operator will reduce to weighted arithmetic average operator of IVLIFNs shown as follows [47]:

$$WIVLIFMSM_{\omega}^{(1)}\left(\hat{\gamma}_1,\hat{\gamma}_2,\ldots,\hat{\gamma}_n\right)=\left(\frac{\bigoplus_{1\le i_1\le n}\bigotimes_{j=1}^{1}n\omega_{i_j}\hat{\gamma}_{i_j}}{C_n^1}\right)^{\frac{1}{1}}=\sum_{i=1}^{n}\omega_i\hat{\gamma}_i.$$
$$=\left(s_{\left[\tau-\tau\prod_{i=1}^{n}\left(\left(1-\frac{k_i}{\tau}\right)^{\omega_i}\right),\tau-\tau\prod_{i=1}^{n}\left(1-\frac{l_i}{\tau}\right)^{\omega_i}\right]},s_{\left[\tau\prod_{i=1}^{n}\left(\frac{p_i}{\tau}\right)^{\omega_i},\tau\prod_{i=1}^{n}\left(\frac{q_i}{\tau}\right)^{\omega_i}\right]}\right) \tag{50}$$

Case 5. When $x=2$, the WIVLIFMSM will reduce to special weighted BM (when $q=1$, $p=1$) operator shown as follows [7]:

$$WIVLIFMSM_{\omega}^{(2)}\left(\hat{\gamma}_1,\hat{\gamma}_2,\ldots,\hat{\gamma}_n\right)=\left(\frac{\bigoplus_{1\le i_1\prec i_2\le n}\bigotimes_{j=1}^{2}n\omega_{i_j}\hat{\gamma}_{i_j}}{C_n^2}\right)^{\frac{1}{2}}=\left(\frac{2}{(n-1)n}\sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(n\omega_i\hat{\gamma}_i^{1}\right)\left(n\omega_j\hat{\gamma}_j^{1}\right)\right)$$
$$=\left(s_{\left[\tau\left(1-\left(\prod_{\substack{i_1=i_2=1\\ i_1\ne i_2}}^{n}\frac{\left(1-\left(1-k_{i_1}\right)^{n\omega_{i_1}}\right)\left(1-\left(1-k_{i_2}\right)^{n\omega_{i_2}}\right)}{\tau}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},\tau\left(1-\left(\prod_{\substack{i_1=i_2=1\\ i_1\ne i_2}}^{n}\frac{\left(1-\left(1-l_{i_1}\right)^{n\omega_{i_1}}\right)\left(1-\left(1-l_{i_2}\right)^{n\omega_{i_2}}\right)}{\tau}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right]},\right. \tag{51}$$
$$\left.s_{\left[\tau-\tau\left(1-\left(\prod_{\substack{i_1=i_2=1\\ i_1\ne i_2}}^{n}1-\left(1-\left(\frac{p_{i_1}}{\tau}\right)^{n\omega_{i_1}}\right)\left(1-\left(\frac{p_{i_1}}{\tau}\right)^{n\omega_{i_1}}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},\tau-\tau\left(1-\left(\prod_{\substack{i_1=i_2=1\\ i_1\ne i_2}}^{n}1-\left(1-\left(\frac{q_{i_1}}{\tau}\right)^{n\omega_{i_1}}\right)\left(1-\left(\frac{q_{i_1}}{\tau}\right)^{n\omega_{i_1}}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right]}\right).$$

Case 6. When $x = n$, the WIVLIFMSM operator will reduce to special weighted operator of IVLIFNs shown as follows:

$$WIVLIFMSM_{\omega}^{(n)}\left(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n\right) = \left(\frac{\underset{1 \le i_1 \prec ... \prec i_n \le n}{\oplus} \overset{n}{\underset{j=1}{\otimes}} n\omega_{i_j}\hat{\gamma}_{i_j}}{C_n^n}\right)^{\frac{1}{n}} = \left(\prod_{i=1}^{n} n\omega_i\hat{\gamma}_i\right)^{\frac{1}{n}}$$

$$\left(s_{\left[\tau\left(\frac{\prod_{i=1}^{n}\tau-\tau\left(1-\frac{k_i}{\tau}\right)^{n\omega_i}}{\tau^n}\right)^{\frac{1}{n}}, t\left(\frac{\prod_{i=1}^{n}\tau-\tau\left(1-\frac{l_i}{\tau}\right)^{n\omega_i}}{\tau^n}\right)^{\frac{1}{n}}\right]}, s_{\left[\tau-\tau\left(\prod_{i=1}^{n}\left(1-\left(\frac{p_i}{\tau}\right)^{n\omega_i}\right)\right)^{\frac{1}{n}}, \tau-\tau\left(\prod_{i=1}^{n}\left(1-\left(\frac{q_i}{\tau}\right)\right)^{n\omega_i}\right)^{\frac{1}{n}}\right]} \right). \tag{52}$$

A new MADM method for IVLIFNs based on the WIVLIFMSM operator

In this part, we would apply the WIVLIFWMSM operator to solve MADM problems with IVLIFNs.

Suppose $X = \{\chi_1, \chi_2, ..., \chi_m\}$ is a group of discrete alternatives, and $C = \{c_1, c_2, ..., c_n\}$ is a group of attributes with the weight vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ which satisfies $\omega_j \in [0, 1]$, $j = 1$ *to* n, and $\sum_{j=1}^{n}\omega_j = 1$. The expert can provide the assessment information of the attributes about alternatives by IVLIFN, and then the interval-valued linguistic intuitionistic fuzzy decision matrix (IVLIFDM) for IVLIFNs $R = \left(\hat{\gamma}_{ij}\right)_{m\times n}$ is constructed, where $\hat{\gamma}_{ij} = \left(s_{\left[k_{ij}, l_{ij}\right]}, s_{\left[p_{ij}, q_{ij}\right]}\right)$ $(i = 1 \ to \ m.j = 1 \ to \ n)$. The goal of the decision-making problem is to give a ranking result for all alternatives.

Firstly, we give the ranking steps based on the WIVLIFMSM operator as follows:

Step 1. Normalize the DM matrix.

Since there are two types for attributes, i.e., benefit type and cost type. We can convert the cost attribute values to benefit ones, and the transformed decision matrix is expressed by $\widehat{R} = \left[\widehat{\gamma}_{ij}\right]_{m\times n}$, $(i = 1 \ to \ m, j = 1 \ to \ n)$, where

$$\widehat{\gamma}_{ij} = \begin{cases} \left(s_{\left[k_{ij}, l_{ij}\right]}, s_{\left[p_{ij}, q_{ij}\right]}\right) & \text{for benefit attribute } C_j \\ \left(s_{\left[p_{ij}, q_{ij}\right]}, s_{\left[k_{ij}, l_{ij}\right]}\right) & \text{for cost attribute } C_j \end{cases}. \tag{53}$$

Step 2. Use the IVLIFWMSM operator.

$$\widehat{\gamma}_i = IVLIFWMSM_{\omega}\left(\widehat{\gamma}_{i1}, \widehat{\gamma}_{i2}, ..., \widehat{\gamma}_{in}\right) \tag{54}$$

to aggregate all attribute values $\widehat{\gamma}_{ij}(j = 1 \ to \ n)$ of each alternative and get the overall values.

Step 3. Calculate score value by Eq. (33).

Step 4. Rank the all alternatives $\{\chi_1, \chi_2, ..., \chi_m\}$ and select the best one.

An illustrative example

In this part, we will give an example to explain the proposed method.

Since we firstly proposed the concept of IVLIFNs, there is no appropriate example which can be cited to this paper. In [26], the attribute values are presented by LIFNs; we made a minor revision and converted them to IVLIFNs.

A manufacturing company wants to build a new subsidiary company in one suitable position, suppose$X = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ is a group of four potential sites as the alternatives and $C = \{c_1, c_2, c_3, c_4, c_5\}$ is a group of

Table 1 Decision matrix R

	c_1	c_2	c_3	c_4	c_5
χ_1	$(s_{[6, 7]}, s_{[1, 1]})$	$(s_{[5, 6]}, s_{[1, 2]})$	$(s_{[4, 5]}, s_{[1, 3]})$	$(s_{[6, 7]}, s_{[1, 1]})$	$(s_{[5, 6]}, s_{[1, 2]})$
χ_2	$(s_{[4, 5]}, s_{[1, 2]})$	$(s_{[5, 7]}, s_{[1, 1]})$	$(s_{[5, 6]}, s_{[1, 2]})$	$(s_{[5, 6]}, s_{[1, 2]})$	$(s_{[6, 7]}, s_{[1, 1]})$
χ_3	$(s_{[5, 6]}, s_{[1, 2]})$	$(s_{[4, 5]}, s_{[2, 3]})$	$(s_{[6, 7]}, s_{[1, 1]})$	$(s_{[5, 6]}, s_{[1, 2]})$	$(s_{[3, 4]}, s_{[3, 4]})$
χ_4	$(s_{[4, 5]}, s_{[2, 3]})$	$(s_{[6, 7]}, s_{[1, 1]})$	$(s_{[4, 5]}, s_{[2, 3]})$	$(s_{[4, 6]}, s_{[1, 2]})$	$(s_{[3, 4]}, s_{[3, 4]})$

Table 2 Scores of $Ivls(\gamma_i)$ by the different parameter value x

x	Score of $Ivls(\hat{\gamma}_i)$	Ranking
$x=1$	$Ivls(\hat{\gamma}_1) = 4.430; Ivls(\hat{\gamma}_2) = 4.416;\ Ivls(\hat{\gamma}_3) = 3.159; Ivls(\hat{\gamma}_4) = 3.100.$	$\chi_1 \succ \chi_2 \succ \chi_3 \succ \chi_4$
$x=2$	$Ivls(\hat{\gamma}_1) = 4.217; Ivls(\hat{\gamma}_2) = 4.139;\ Ivls(\hat{\gamma}_3) = 3.007; Ivls(\hat{\gamma}_4) = 2.419.$	$\chi_1 \succ \chi_2 \succ \chi_3 \succ \chi_4$
$x=3$	$Ivls(\hat{\gamma}_1) = 4.129; Ivls(\hat{\gamma}_2) = 3.954;\ Ivls(\hat{\gamma}_3) = 2.929; Ivls(\hat{\gamma}_4) = 2.496.$	$\chi_1 \succ \chi_2 \succ \chi_3 \succ \chi_4$
$x=4$	$Ivls(\hat{\gamma}_1) = 4.021; Ivls(\hat{\gamma}_2) = 3.990;\ Ivls(\hat{\gamma}_3) = 2.873; Ivls(\hat{\gamma}_4) = 2.158.$	$\chi_1 \succ \chi_2 \succ \chi_3 \succ \chi_4$
$x=5$	$Ivls(\hat{\gamma}_1) = 3.950; Ivls(\hat{\gamma}_2) = 3.954;\ Ivls(\hat{\gamma}_3) = 2.824; Ivls(\hat{\gamma}_4) = 2.091.$	$\chi_2 \succ \chi_1 \succ \chi_3 \succ \chi_4$

attributes, where $(c_1, c_2, c_3, c_4, c_5)$ stand for "the price of land," "the distance of sale market," "the distance of resources," "the labor market," and "local economy," respectively, and the weight of attributes is$w = (0.2, 0.25, 0.15, 0.18, 0.22)^T$. The expert evaluates each alternative $\chi_i(1, ..., 4)$ with respect to each attribute$c_j(j = 1, 2, 3, 4, 5)$ based on the LTS: $S = \{s_0 =$ extremely bad, $s_1 =$ very bad, $s_2 =$ bad, $s_3 =$ slightly bad, $s_4 =$ fair, $s_5 =$ slightly good, $s_6 =$ good, $s_7 =$ very good, $s_8 =$ extremely good$\}$ and can give the evaluation value $\tilde{\gamma}_{ij}$ which is expressed by the IVLIFN, then the decision matrix $R = \left(\tilde{\gamma}_{ij}\right)_{4\times5}$ is constructed and shown in Table 1.

Next, we will present the procedure of decision-making based on the WIVLIFMSM operator.

Step 1. Normalize the decision-making matrix.

Because all attributes are benefit type, this step can be omitted.

Step 2. Use the WIVLIFMSM operator to produce the overall values $\gamma_i (i = 1, 2, 3, 4)$ of each alternative (suppose$x$ $= 2$), then we can get

$\gamma_1 = \left(s_{[5.152, 6.183]}, s_{[1.083, 1.817]}\right)$,
$\gamma_2 = \left(s_{[4.916, 6.129]}, s_{[1.083, 1.684]}\right)$,
$\gamma_3 = \left(s_{[4.509, 5.634]}, s_{[1.623, 2.436]}\right)$,
and $\gamma_4 = \left(s_{[4.102, 5.303]}, s_{[1.868, 2.697]}\right)$.

Step 3. Calculate the score values of $\gamma_i (i = 1\ to\ 4)$, we can get

$Ivls(\gamma_1) = 4.217;\quad Ivls(\gamma_2) = 4.139;\quad Ivls(\gamma_3) = 3.007;$ $Ivls(\gamma_4) = 2.419.$.

Step 4. Rank all alternatives based on the score values, we get $\chi_1 \succ \chi_2 \succ \chi_3 \succ \chi_4$.

So, the best choice is χ_1.

Table 3 The comprehensive values of decision-making matrix aggregated by Chen in [10]

	de_1	de_2	de_3	de_4
χ_1	$(s_{6.199}, s_{1.578})$	$(s_{5.458}, s_{2.363})$	$(s_{5.598}, s_{1.647})$	$(s_{5.129}, s_{2.023})$
χ_2	$(s_{6.138}, s_{1.444})$	$(s_{5.715}, s_{1.433})$	$(s_{6.343}, s_{1.301})$	$(s_{6.127}, s_{1.133})$
χ_3	$(s_{5.428}, s_{1.690})$	$(s_{5.501}, s_{1.966})$	$(s_{4.673}, s_{1.842})$	$(s_{4.572}, s_{2.471})$
χ_4	$(s_{5.510}, s_{1.902})$	$(s_{5.644}, s_{2.093})$	$(s_{5.846}, s_{1.712})$	$(s_{4.871}, s_{1.927})$

Further, we can obtain the aggregation results from the different parameter x, which are shown in Table 2.

From Table 2, the scores of the aggregation results are different for the different parameter x, and then the ranking results are different. When parameter $x = 1, 2, 3, 4$, the ranking results are all $\chi_1 \succ \chi_2 \succ \chi_3 \succ \chi_4$. When $x = 1$, the IVLIFWMSM operator will reduce to weighted arithmetic average operator of IVLIFNs; the best choice is χ_1, when parameter $x = 5$; the IVLIFWMSM operator will reduce to weighted geometric operator of IVLIFNs; the ranking result is $\chi_2 \succ \chi_1 \succ \chi_3 \succ \chi_4$, and the best choice is χ_2. Obviously, as parameter x increases, we can consider the more interrelationships, and the ranking results will be changed. So, in the real decision-making, we can assign the parameter value x according to the actual needs.

Further, we can give a comparison with the other method. Since there is no decision method based on IVLIFNs, it is difficult to verify the feasible and effectiveness of the proposed method in this paper. However, we can use the special case of IVLIFWMSM operator, i.e., when the upper and lower limits of the membership function and non-membership function in IVLIFNs are equal respectively, and the IVLIFNs will reduce to LIFNs, and the WIVLIFMSM operator will reduce to LIFWMSM operator, then we can compare it with the method in [10].

In [10], $X = \{\chi_1, \chi_2, \chi_3, \chi_4\}$ is a group of four alternatives which stand for potential global suppliers under consideration and $A = \{a_1, a_2, a_3, a_4, a_5\}$ is a group of five attributes, where $a_j(j = 1\ to\ 5)$ represents "overall cost of the product," "quality of the product," "service performance of supplier," "supplier's profile," and "risk factor," respectively. The weight vector of all attributes is $\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)^T = (0.25, 0.2, 0.15, 0.18, 0.22)^T$, and four decision-makers $de_k(k = 1, 2, ..., 4)$ can evaluate the four alternatives under the five attributes and construct the decision matrices based on IVLIFNs; the goal is to select the best supplier.

Chen [10] use the LIFWA operator to aggregate the whole attributes of each alternative and get a comprehensive decision matrix, the results are presented in Table 3.

Firstly, we extend the LIFNs which are presented in Table 3 to IVLIFNs, for example, $(s_{6.199}, s_{1.578})$ can be extended to IVLIFNs $(s_{[6.199,\,6.199]}, s_{[1.578,\,1.578]}) \in \Gamma_{[0,\,8]}$, and the decision

Table 4 The decision-making matrix of IVLIFNs extended from Table 3

	de_1	de_2	de_3	de_4
χ_1	$(s_{[6.199, 6.199]}, s_{[1.578, 1.578]})$	$(s_{[5.458, 5.458]}, s_{[2.363, 2.363]})$	$(s_{[5.598, 5.589]}, s_{[1.647, 1.647]})$	$(s_{[5.129, 5.129]}, s_{[2.023, 2.023]})$
χ_2	$(s_{[6.138, 6.138]}, s_{[1.444, 1.444]})$	$(s_{[5.715, 5.715]}, s_{[1.433, 1.433]})$	$(s_{[6.343, 6.343]}, s_{[1.301, 1.301]})$	$(s_{[6.127, 6.127]}, s_{[1.133, 1.133]})$
χ_3	$(s_{[5.428, 5.428]}, s_{[1.690, 1.690]})$	$(s_{[5.501, 5.501]}, s_{[1.966, 1.966]})$	$(s_{[4.673, 4.673]}, s_{[1.842, 1.842]})$	$(s_{[4.572, 4.572]}, s_{[2.471, 2.471]})$
χ_4	$(s_{[5.510, 5.510]}, s_{[1.902, 1.902]})$	$(s_{[5.644, 5.644]}, s_{[2.093, 2.093]})$	$(s_{[5.846, 5.846]}, s_{[1.712, 1.712]})$	$(s_{[4.871, 4.871]}, s_{[1.927, 1.927]})$

Table 5 Score results with different parameter x

x	Score of $Ivls(\hat{\chi}_i)$	Ranking
$x = 1$	$Ivls(\hat{\chi}_1) = 3.792; Ivls(\hat{\chi}_2) = 4.809;\ Ivls(\hat{\chi}_3) = 3.066; Ivls(\hat{\chi}_4) = 3.619.$	$\chi_2 \succ \chi_1 \succ \chi_4 \succ \chi_3$
$x = 2$	$Ivls(\hat{\chi}_1) = 3.463; Ivls(\hat{\chi}_2) = 4.599;\ Ivls(\hat{\chi}_3) = 2.870; Ivls(\hat{\chi}_4) = 3.363.$	$\chi_2 \succ \chi_1 \succ \chi_4 \succ \chi_3$
$x = 3$	$Ivls(\hat{\chi}_1) = 3.575; Ivls(\hat{\chi}_2) = 4.639;\ Ivls(\hat{\chi}_3) = 2.964; Ivls(\hat{\chi}_4) = 3.448.$	$\chi_2 \succ \chi_1 \succ \chi_4 \succ \chi_3$
$x = 4$	$Ivls(\hat{\chi}_1) = 3.533; Ivls(\hat{\chi}_2) = 4.608;\ Ivls(\hat{\chi}_3) = 2.946; Ivls(\hat{\chi}_4) = 3.420.$	$\chi_2 \succ \chi_1 \succ \chi_4 \succ \chi_3$

matrix of IVLIFNs had been normalized obviously, and the results are presented in Table 4.

Next, we use the IVLIFWMSM operator to aggregate the fuzzy information in Table 4 for each alternative and get the score value and rank all alternatives, and then we present the results in Table 5 and Fig. 1.

From Table 5 and Fig. 1, we can know that whatever the parameter x is, the ranking results are the same, and the best alternative is χ_2.

Clearly, in this paper, the ranking results are completely the same as the method [10], and the best alternatives are all χ_2. So, it can prove the method in this paper is effective.

In this paragraph, *lvls1* represents $Ivls(\hat{\chi}_1)$,…. *Lvls4* represents $Ivls(\hat{\chi}_4)$.

Compared with the method by [10], we can find that the method in [10] can aggregate fuzzy information more easily, but it supposes that the input arguments are independent and it does not consider the interrelationship among the input arguments. But the new proposed operators in this paper can consider the interrelationships among the arguments by the parameter value x, and they can make the aggregating result more accurate; meanwhile, they can reduce the information loss more effectively. Thus, the new proposed method is more general and flexible to solve MADM problems, especially when the decision-makers should consider the interrelationship among the input parameters.

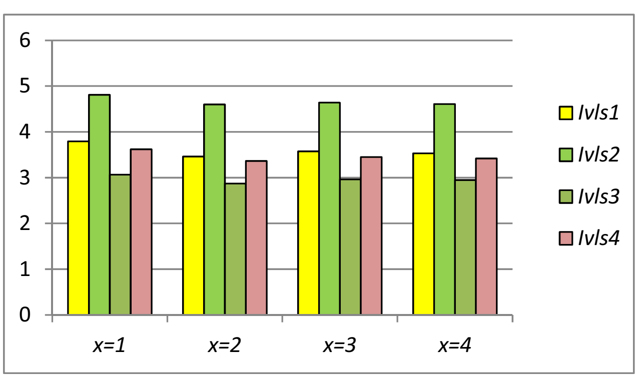

Fig. 1 Score results for different parameter x

Conclusions

Since the IVIFNs can express the fuzzy information more conveniently and the LIFNs can describe the qualitative information easier, in this paper, we proposed the concept of IVLIFNs which are suited for the decision-makers' cognition and gave some operations based on IVLIFNs. Further, we defined the score value and the accuracy value, and then gave some properties. Subsequently, we developed the IVLIFMSM operator and the WIVLIFMSM operator and proved some useful properties. Furthermore, we presented an MADM method with IVLIFNs in detail and gave an example to demonstrate the practicality and effectiveness of the proposed method. Finally, we gave a comparison with the other method to testify our method. In a word, because of the particularity of human cognition activity and complexity of the decision-making environment, the new proposed concept of IVLIFNs is more suitable to depict the complex fuzzy information. In addition, because the proposed method can consider the interrelationship among multiple input arguments by the parameter value x, and also consider the cognition of decision-makers, they can make the decision-making results more reasonable and accurate.

In further research, we can apply the IVLIFWMSM operator to real decision-making problems such as pattern recognition, fuzzy cluster analysis, cognitive computation, and uncertain programming, etc. In addition, we can also develop some other aggregation operators based on IVLIFNs for future research.

Funding information This paper is supported by the National Natural Science Foundation of China (Nos. 71771140 and 71471172), the Special Funds of Taishan Scholars Project of Shandong Province (No. ts201511045), and Shandong Provincial Social Science Planning Project (Nos. 17BGLJ04, 16CGLJ31 and 16CKJJ27).

Compliance with Ethical Standards

Conflicts of Interest The authors declare that they have no conflict of interest.

Research Involving Human Participants and/or Animals This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Abu-Saris R, Hajja M. On Gauss compounding of symmetric weighted arithmetic means. J Math Anal Appl. 2006;322:729–34.
2. Atanassov KT. Intuitionistic fuzzy sets. Fuzzy Sets Syst. 1986;20:87–96.
3. Atanassov KT. More on intuitionistic fuzzy sets. Fuzzy Sets Syst. 1989;33:37–46.
4. Atanassov KT. Operators over interval-valued intuitionistic fuzzy sets. Fuzzy Sets Syst. 1994;64:159–74.
5. Atanassov KT, Gargov G. Interval-valued intuitionistic fuzzy sets. Fuzzy Sets Syst. 1989;31(3):343–9.
6. Bapat RB. Symmetric function means and permanents. Linear Algebra Appl. 1993;182(101–8.
7. Beliakov G, James S. On extending generalized Bonferroni means to Atanassov orthopairs in decision making contexts. Fuzzy Sets Syst. 2013;211(84–9.
8. Bonferroni C. Sulle medie multiple di potenze. Boll Mat Ital. 1950;5:267–70.
9. Chen SM, Lee LW, Liu HC, Yang SW. Multi-attribute decision making based on interval-valued intuitionistic fuzzy values. Expert Syst Appl. 2012;39:10343–51.
10. Chen ZC, Liu PH. An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. Int J Comput Intell Syst. 2015;8(4):747–60.
11. Delgado M, Herrera F, Herrera-Viedma E, Martinez L. Combining numerical and linguistic information in group decision making. Inf Sci. 1998;107:177–94.
12. Detemple D, Robertson J. On generalized symmetric means of two variables. Univ Beograd Publ Elektrotehn FakSer Mat Fiz. 1979;677(634):236–8.
13. Farhadinia B, Xu Z. Distance and aggregation-based methodologies for hesitant fuzzy decision making. Cogn Comput. 2017;9(1):81–94.
14. Gao P. On a conjecture on the symmetric means. J Math Anal Appl. 2008;337:416–24 puter and System Sciences 70(1) (2005)73–85.
15. Herrera F, Herrera-Viedma E, Martinez L. A fusion approach for managing multi-granularity linguistic term sets in decision-making. Fuzzy Sets Syst. 2000;114(1):43–58.
16. Herrera F, Herrera-Viedma E. Aggregation operators for linguistic weighted information. IEEE Trans Syst Man Cybern Part A Syst Hum. 1997;27(5):646–56.
17. Liu HZ, Pei DW. HOWA operator and its application to multi-attribute decision making. J Zhejiang Sci Technol Univ. 2012;25:138–42.
18. Liu PD. Some generalized dependent aggregation operators with intuitionistic linguistic numbers and their application to group decision making. J Comput Syst Sci. 2013;79(1):131–43.
19. Liu PD. The multi-attribute group decision making method based on the interval grey linguistic variables weighted aggregation operator. J Intell Fuzzy Syst. 2013;24(2):405–14.
20. Liu PD. Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. IEEE Trans Fuzzy Syst. 2014;22(1):83–97.
21. Liu PD, Chen YB, Chu YC. Intuitionistic uncertain linguistic weighted Bonferroni OWA operator and its application to multiple attribute decision making. Cybern Syst. 2014;45(5):418–38.
22. Liu PD, Li Y. Multiple attribute group decision making methods based on trapezoidal fuzzy two-dimension linguistic power generalized aggregation operators. Soft Comput. 2016;20(7):2689–704.
23. Liu PD, Li HG. Multiple attribute decision making method based on some normal neutrosophic Bonferroni mean operators. Neural Comput & Applic. 2017;28(1):179–94.
24. Liu PD, Li HG. Interval-valued intuitionistic fuzzy power Bonferroni aggregation operators and their application to group decision making. Cogn Comput. 2017;9:494–512. https://doi.org/10.1007/s12559-017-9453-9.
25. Liu PD, Liu ZM, Zhang X. Some intuitionistic uncertain linguistic Heronian mean operators and their application to group decision making. Appl Math Comput. 2014;230:570–86.
26. Liu PD, Qin XY. Power average operators of linguistic intuitionistic fuzzy numbers and their application to multiple-attribute decision making. J Intell Fuzzy Syst. 2017;01:1029–43.
27. Liu PD, Shi LL. Some neutrosophic uncertain linguistic number Heronian mean operators and their application to multi-attribute group decision making. Neural Comput & Applic. 2016;28:1079–93. https://doi.org/10.1007/s00521-015-2122-6.
28. Liu PD, Tang GL. Multi-criteria group decision-making based on interval neutrosophic uncertain linguistic variables and Choquet integral. Cogn Comput. 2016;8(6):1036–56.
29. Liu PD, Wang YM. Multiple attribute group decision making methods based on intuitionistic linguistic power generalized aggregation operators. Appl Soft Comput. 2014;17(90–104.
30. Liu PD, Wang YM. Multiple attribute decision-making method based on single valued Neutrosophic normalized weighted Bonferroni mean. Neural Comput & Applic. 2014;25(7–8):2001–10.
31. Maclaurin C. A second letter to Martin Folkes, Esq; concerning the roots of equations, with demonstration of other rules of algebra. Philos Trans R Soc Lond Ser A. 1729;36:59–96.
32. Pecaric J, Wen JJ, Wang WL, Lu T. A generalization of Maclaurin's inequalities and its applications. Math Inequal Appl. 2005;8:583–98.
33. Qin JD, Liu XW. An approach to intuitionistic fuzzy multiple attribute decision making based on Maclaurin symmetric mean operators. J Intell Fuzzy Syst. 2014;27:2177–90.
34. Qin JD, Liu XW. Approaches to uncertain linguistic multiple attribute decision making based on dual Maclaurin symmetric mean. J Intell Fuzzy Syst. 2015;29:171–86.
35. Qin JD, Liu XW. Hesitant fuzzy Maclaurin symmetric mean operators and its application to multiple-attribute decision making. Int J Fuzzy Syst. 2015;17(4):509–20.
36. Tian Z, Wang J, Wang JQ, Zhang HY. A likelihood-based qualitative flexible approach with hesitant fuzzy linguistic information. Cogn Comput. 2016;8(4):670–83.
37. Wang JQ, Kuang JJ, Wang J, et al. An extended outranking approach to rough stochastic multi-criteria decision-making problems. Cogn Comput. 2016;8(6):1144–60.

38. Wang WZ, Liu XW. Intuitionistic fuzzy information aggregation using Einstein operations. IEEE Trans Fuzzy Syst. 2012;20(5): 923–38.
39. Xu ZS. A method based on linguistic aggregation operators for group decision making with linguistic preference relations. Inf Sci. 2004;166:19–30.
40. Xu ZS. Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment. Inf Sci. 2004;168:171–84.
41. Xu ZS. Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst. 2007;15(6):1179–87.
42. Xu ZS, Yager RR. Some geometric aggregation operators based on intuitionistic fuzzy sets. Int J Gen Syst. 2006;35:417–33.
43. Yao YY. Three-way decisions and cognitive computing. Cogn Comput. 2016;8(4):543–54.
44. Yu DJ. Group decision making based on generalized intuitionistic fuzzy prioritized geometric operator. Int J Intell Syst. 2012;27:635–61.
45. Zadeh LA. Fuzzy sets. Inf Control. 1965;8:338–56.
46. Zadeh LA. The concept of a linguistic variable and its applications to approximate reasoning. Inf Sci. 1975;8(3):199–249.
47. Zhao H, Xu ZS, Ni MF, Liu S. Generalized aggregation operators for intuitionistic fuzzy sets. Int J Intell Syst. 2010;25:1–30.
48. Zhao N, Xu Z, Liu F. Group decision making with dual hesitant fuzzy preference relations. Cogn Comput. 2016;8(6):1119–43.