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Stochastic Multiple-Attribute Decision Making Method Based on Stochastic Dominance and Almost Stochastic Dominance Rules with an Application to Online Purchase Decisions

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Abstract

Although some stochastic multiple-attribute decision making (SMADM) methods based on the stochastic dominance (SD) rules have been proposed, there is still the limitation that the dominance relations between some pairs of alternatives cannot be identified. In this paper, almost stochastic dominance (ASD) rules are used as supplements of the SD rules to identify dominance relations between the pairs of alternatives, and a new method for SMADM based on the SD and ASD rules is proposed. In the method, a procedure for identifying the dominance relation between each pair of alternatives based on the SD and ASD rules is given. Then, according to the identified dominance relation, the priority degree that one alternative is superior to another alternative concerning each attribute is calculated. Further, according to the obtained priority degrees, an approach for ranking alternatives is proposed using the simple weighted method. Finally, the proposed method is applied to the selection of passenger car(s) based on online ratings, and a comparison between the proposed method and the existing methods based on a numerical example is given. The proposed method can obtain more precise ranking results of alternatives. ASD rules are important supplements of the SD rules for identifying dominance relations. Based on the SD and ASD rules, the proposed SMADM method is important for developing theories and methods for SMADM.

Keywords Stochastic multiple-attribute decision making (SMADM) · Stochastic dominance (SD) · Almost stochastic dominance (ASD) · Alternative ranking

Introduction

In reality, there are many multiple-attribute decision making (MADM) problems, i.e., problems of selecting alternatives associated with multiple attributes [6]. Many models and methods have been developed to solve MADM problems in which attribute values can be in different formats, such as crisp numbers, interval numbers, linguistic terms [15], or Z-numbers [28]. In some practical MADM problems, the values of some or all attributes may be in the form of stochastic variables [2, 3, 8, 14, 19, 29–32, 37]. For example, in the problem of selecting a passenger car based on online product ratings, attribute values are in the form of stochastic variables because the statistical results of online ratings provided by a large number of consumers follow probability distributions [16]. In large group decision making (LGDM) problems, the collective evaluations of alternatives concerning attributes can be regarded as stochastic variables with percentage distributions [17, 18]. In the problem of selecting the most desirable strategy for an electricity retailer, the consequences of alternatives concerning four attributes (i.e., long-term profits, short-term profits, market share, and green market share) may be stochastic variables with normal probability distributions [10]. Therefore, the MADM problem in which attribute values are in the form of stochastic variables, i.e., the stochastic MADM (thereafter SMADM) problem, is a noteworthy research topic with extensive theoretical and practical backgrounds.

So far, many models and methods have been proposed to solve SMADM problems from different perspectives, such as the expected utility [9], the confidence indices [20, 21], the preference indices [1, 3, 19], and the stochastic dominance (SD) rules

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[33]. Additionally, some artificial intelligence methods [4, 5] have the potential to be used for solving the SMADM problems.

In the existing studies, the methods based on the SD rules are regarded as the ones with higher practical significance since less restrictive assumptions on the utility functions of decision-makers (DMs) are needed. The methods based on the SD rules commonly consist of two processes: comparison and selection. The former identifies SD relations among alternatives using the SD rules. The latter uses an outranking procedure, such as ELECTRE (ELimination Et Choix Traduisant la REalité) and PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluation), to determine a ranking of alternatives based on the identified SD relations [7, 8, 22–26, 34–36, 38]. For example, Zaras [36] proposed a method for solving the SMADM problem with deterministic, fuzzy, and stochastic evaluation information. In the method, based on the SD rules, the concept of mixed-data multiattribute dominance is first proposed. Then, using the mixed-data multiattribute dominance, the dominance relation between each pair of alternatives concerning each attribute is identified. Further, according to the identified dominance relations, the rough set approach is employed to determine the ranking of the alternatives. Nowak [24] proposed a method based on the SD rules to solve the SMADM problem. In the method, the dominance relation between each pair of alternatives concerning each attribute is determined based on the SD rules. Then, based on the determined dominance relation and a predefined preference threshold, the strict preference relation, the weak preference relation, or the indifference preference relation between each pair of alternatives concerning each attribute is determined. Further, based on the determined preference relations, the ranking of all alternatives is determined using the ELECTRE-III outranking procedure. Nowak [26] proposed an SMADM method based on the SD rules, the interactive approach and the preference threshold. The main idea of the method is to progressively reduce the set of alternatives using a multiple round interactive procedure. In each round of the interactive procedure, several alternatives are weeded out according to the SD relations and the preference threshold concerning each attribute. If the DM is able to make a final choice, then the procedure ends; otherwise, the DM is asked to adjust the preference threshold or the aspiration level concerning different attributes. Zhang et al. [38] proposed a method based on the SD degrees (SDD) to solve the SMADM problem. In the method, the concept of SDD is first introduced, and the computation formula of SDD is given. Then, the SDD matrix of the pairwise comparisons of the alternatives concerning each attribute is constructed. Further, the overall SDD matrix is obtained by aggregating the SDD matrices concerning multiple attributes and the attribute weights. Finally, according to the overall SDD matrix, the ranking of all alternatives is determined using the PROMETHEE-II method.

It is necessary to point out that when the SD rules are used, the dominance relations between some pairs of alternatives cannot be identified. This would lead to the absences of some dominance relations or lead to the unclear rankings of alternatives, i.e., the ranking orders of some alternatives cannot be distinguished. The concept of almost stochastic dominance (ASD) is first proposed by Leshno and Levy [11]. The core idea of the ASD is to establish rules that are used to formally reveal a preference for "most" DMs, but not for "all" of them, by eliminating some extreme utility functions to relax the strict restrictions on the distribution functions [11]. However, Leshno and Levy [11] just put forward the concept of ASD, it is seldom seen that the concept of ASD is used to solve practical decision-making problems. Obviously, if the ASD rules can be used as supplements of the SD rules to identify dominance relations between the pairs of alternatives, then it is necessary to develop a new SMADM method based on the SD rules and the ASD rules.

The objective of this paper is to develop a novel SMADM method based on the SD rules and the ASD rules. First, according to concepts of the SD and the ASD, definitions of the SD rules and the ASD rules are introduced, and a theoretical analysis of the properties of the SD and ASD rules is given. Then, based on the SD and ASD rules, a procedure for identifying the dominance relation between each pair of alternatives is given. Further, according to the identified dominance relations, a priority function with respect to each attribute is constructed to measure the priority degrees of the pairwise comparisons of alternatives. Moreover, using the simple weighted method, an overall priority degree for each pair of alternatives is calculated, and a priority degree matrix of the pairwise comparisons of alternatives is built. Finally, according to the priority degree matrix, an approach for ranking alternatives is given. In the approach, two vectors are constructed according to the maximum values in each column and each row in the priority degree matrix, and the closeness coefficient of each alternative is determined by calculating the degree of deviation. The ranking of alternatives can be determined according to the obtained closeness coefficients of alternatives.

This study contributes to the existing literature in three aspects. First, it is a good attempt to solve the SMADM by using both the SD and the ASD rules. Second, compared with the existing methods, the method proposed in this paper can be used to identify the dominance relation between each pair of alternatives more precisely. Third, the ranking of alternatives obtained by the proposed method would be more consistent with human cognition since both the SD and the ASD rules for comparing stochastic variables embody human cognitive characteristics.

The rest of this paper is arranged as follows. In "SD and ASD Rules" section, the definitions of the SD rules and the ASD rules are introduced, and the theoretical analysis of the properties of the SD and ASD rules is given. In "The Proposed SMADM Method" section, a method for SMADM based on the SD and ASD rules is presented. In "An Application to Support Online Purchasing Decision" section, an example of the selection of passenger car(s) based on online ratings is given to illustrate the application of the proposed method. In "Comparative Analysis" section, a comparison between the proposed method and the existing methods based on a numerical example is given to illustrate the feasibility and the validity of the proposed method. Finally, "Conclusions" section summarizes and highlights the main features of this paper.

SD and ASD Rules

SD and ASD are the axiomatic rules for comparing stochastic variables. The SD rules include the first-degree, second-degree, and third-degree SD rules, and the ASD rules include the first-degree, second-degree, and third-degree ASD rules. In this section, the definitions of the SD rules and the ASD rules are introduced, and the theoretical analysis of the properties of the SD and ASD rules is given.

Let a and b be two real numbers such that $a < b$. Let X and Y be two stochastic variables on interval $[a, b]$ and $F(x)$ and $G(x)$ be cumulative distribution functions of X and Y, respectively, such that $F(a) = G(a) = 0$ and $F(b) = G(b) = 1$. Let $E_F(X)$ and $E_G(Y)$ be the expected values of X and Y, respectively. The SD rules, i.e., the definitions of the first-degree, the second-degree, and the third-degree SD rules, are given below [36].

Definition 1. $F(x)$ stochastically dominates $G(x)$ by the first degree (noted as $F(x)\,\text{FSD}\,G(x)$) if and only if $F(x) \neq G(x)$ and $H_1(x) = F(x) - G(x) \leq 0$, $x \in [a, b]$.

Definition 2. $F(x)$ stochastically dominates $G(x)$ by the second degree (noted as $F(x)\,\text{SSD}\,G(x)$) if and only if $F(x) \neq G(x)$ and $H_2(x) = \int_a^x H_1(y)dy \leq 0$, $x \in [a, b]$.

Definition 3. $F(x)$ stochastically dominates $G(x)$ by the third degree (noted as $F(x)\,\text{TSD}\,G(x)$) if and only if $F(x) \neq G(x)$ and $H_3(x) = \int_a^x H_2(y)\,dy \leq 0$, $E_F(X) \geq E_G(Y)$, $x \in [a, b]$.

Let $\Omega_1 = \{x|\, G(x) < F(x),\ x \in [a, b]\}$, $\Omega_2 = \left\{x|\int_a^x G(t)\ dt < \int_a^x F(t)\ dt,\ \ x \in \Omega_1\right\}$ and $\Omega_3 = \left\{x|\int_a^x\int_a^x G(t)\,dt\,dt < \int_a^x\int_a^x\ F(t)\,dt\ dt,\ \ x \in \Omega_2\right\}$ be the three subsets in interval $[a, b]$, $x \in [a, b]$. According to the research result of Leshno and Levy [11], the ASD rules, i.e., the definitions of the almost first-degree, the almost second-degree, and the almost third-degree SD rules, are given below.

Definition 4. $F(x)$ stochastically dominates $G(x)$ by the almost first degree (noted as $F(x)\,\text{AFSD}\,G(x)$) if and only if $0 \leq \varepsilon_1 < 0.5$ and $F(x) \neq G(x)$, where $\varepsilon_1 = \int_{\Omega_1}[F(x) - G(x)]\,dx / \int_a^b |F(x) - G(x)|\,dx$.

Definition 5. $F(x)$ stochastically dominates $G(x)$ by the almost second degree (noted as $F(x)$ ASSD $G(x)$) if and only if $0 \leq \varepsilon_2 < 0.5$, $F(x) \neq G(x)$, and $E_F(X) \geq E_G(Y)$, where $\varepsilon_2 = \int_{\Omega_2}[F(x) - G(x)]\,dx / \int_a^b |F(x) - G(x)|\,dx$.

Definition 6. $F(x)$ stochastically dominates $G(x)$ by the almost third degree (noted as $F(x)\,\text{ATSD}\,G(x)$) if and only if $0 \leq \varepsilon_3 < 0.5$, $F(x) \neq G(x)$, and $E_F(X) \geq E_G(Y)$, where $\varepsilon_3 = \int_{\Omega_3}[F(x) - G(x)]\,dx / \int_a^b |F(x) - G(x)|\,dx$.

According to the above definitions, it is easy for us to get the following conclusions.

Remark 1

1) For the first-degree, second-degree, and third-degree SD rules shown in Definitions 1–3, if there is $F(x)$ FSD $G(x)$, $F(x)$ SSD $G(x)$, or $F(x)$ TSD $G(x)$, then there is no $G(x)$ FSD $F(x)$, $G(x)$ SSD $F(x)$, or $G(x)$ TSD $F(x)$.
2) For the almost first-degree, second-degree and third-degree SD rules shown in Definitions 4–6, if there is $F(x)$ AFSD $G(x)$, $F(x)$ ASSD ,$G(x)$, or $F(x)$ ATSD $G(x)$, then there is no $G(x)$ AFSD $F(x)$, $G(x)$ ASSD $F(x)$, or $G(x)$ ATSD $F(x)$.
3) The smaller the value of ε_1, ε_2, or ε_3 is, the greater the degree that $F(x)$ almost stochastically dominants $G(x)$ will be.
4) Consider two arbitrary alternatives A and B in decision analysis. The evaluations of the two alternatives are stochastic variables with cumulative distribution functions $F(x)$ and $G(x)$, respectively. If $F(x)\ \overline{\text{SD}}\ G(x)$, then $A\ \overline{\text{SD}}\ B$, where $\overline{\text{SD}}$ denotes the stochastic dominance relation between alternatives A and B, and $\overline{\text{SD}} \in \{\text{FSD, SSD, TSD, AFSD, ASSD, ATSD}\}$.
5) In decision analysis, even if there is no FSD, SSD, and TSD between some pairs of alternatives, there may still be an AFSD, ASSD, or ATSD. Thus, the almost first-degree, almost second-degree, and almost third-degree SD rules can be used as supplements of the first-, second-, and third-degree SD rules for identifying the SD relations between some pairs of alternatives.

In the following, an example is given to illustrate 5 of Remark 1. Let us consider an investment project selection problem. The DM needs to select one from two investment projects A and B. The gains and winning probabilities of the two projects A and B are uncertain, which are shown in Table 1. Let $F(x)$ and $G(x)$ denote the cumulative distribution functions of the gains of projects A and B, respectively. The curves of $F(x)$ and $G(x)$ are shown in Fig. 1. It can be seen from Fig. 1 and Definitions 1–3 that there is not a first-degree, second-degree, or third-degree SD relation between $F(x)$ and $G(x)$, i.e., there is not an SD relation between projects A and B. Nevertheless, it can be seen from Fig. 1 that the expected values of projects A and B are $E(A)=1 \times 0.1 + 1{,}000{,}000 \times 0.9 = 900000.1$ and $E(B)=2 \times 0.1 + 3 \times 0.9 = 2.9$, respectively. Thus, the DM would select project A because $E(A)$ is significantly greater than $E(B)$. Therefore, there is actually a dominance relationship between the two projects. According to Definitions 4–6, it can be seen that the almost SD relation exists between projects A and B.

According to Definitions 1–6 and the existing studies [11–13], the following properties can be easily found.

Property 1

1) $F(x)$ FSD $G(x)$ if and only if $F(x)$ AFSD $G(x)$ and $\varepsilon_1 = 0$;
2) $F(x)$ SSD $G(x)$ if and only if $F(x)$ ASSD $G(x)$ and $\varepsilon_2 = 0$;
3) $F(x)$ TSD $G(x)$ if and only if $F(x)$ ATSD $G(x)$ and $\varepsilon_3 = 0$.

Proof First, we prove conclusion 1. If $F(x)$ AFSD $G(x)$ and $\varepsilon_1 = 0$, then according to Definition 4, we have $\int_{\Omega_1}[F(x)-G(x)]\,dx = 0$ and $\Omega_1 = \{x|\, G(x) < F(x),\ x \in [a,b]\} = \varnothing$. Since Ω_1 is an empty set, $G(x) \ge F(x)$ follows for any $x \in [a,b]$, i.e., $F(x) - G(x) \le 0$ for any $x \in [a,b]$. Because $F(x) - G(x) \le 0$ for any $x \in [a,b]$ and $F(x) \ne G(x)$, then we know $F(x)$ FSD $G(x)$ according to Definition 1. On the other hand, if $F(x)$ FSD $G(x)$, according to Definition 1, we know $H_1(x) = F(x) - G(x) \le 0$ follows for any $x \in [a,b]$ and $F(x) \ne G(x)$, and thus, $\Omega_1 = \{x|\, G(x) < F(x), x \in [a,b]\} = \varnothing$, i.e., $\varepsilon_1 = \int_{\Omega_1}[F(x)-G(x)]\,dx / \int_a^b |F(x)-G(x)|\,dx = 0$. Therefore, according to Definition 4, we know $F(x)$ AFSD $G(x)$. Similarly, conclusions 2) and 3) can also be proved.

Property 2

1) If $F(x)$ FSD $G(x)$, then $F(x)$ SSD $G(x)$, $F(x)$ TSD $G(x)$, $F(x)$ AFSD $G(x)$, $F(x)$ ASSD $G(x)$, and $F(x)$ ATSD $G(x)$; 2) If $F(x)$ SSD $G(x)$, then $F(x)$ TSD $G(x)$, $F(x)$ ASSD $G(x)$, and $F(x)$ ATSD $G(x)$; and 3) If $F(x)$ TSD $G(x)$, then $F(x)$ ATSD $G(x)$.

Proof First, we prove the conclusion 1. If $F(x)$ FSD $G(x)$, according to Definition 1, we know $H_1(x) = F(x) - G(x) \le 0$ and $F(x) \ne G(x)$. Meanwhile, we also know $E_F(X) \ge E_G(Y)$. Thus, we have $H_2(x) = \int_a^x H_1(y)dy \le 0$ for any $x \in [a,b]$. According to Definition 2, we know $F(x)$ SSD $G(x)$. Further, we know $H_3(x) = \int_a^x H_2(y)\,dy \le 0$ for any $x \in [a,b]$ since $H_2(x) \le 0$, and, according to Definition 3, we know $F(x)$ TSD $G(x)$. According to the above analysis and Property 1, we have $F(x)$ AFSD $G(x)$, $F(x)$ ASSD $G(x)$ and $F(x)$ ATSD $G(x)$. In a similar way, conclusions 2) and 3) can also be proved.

Table 1 Gains and winning probabilities of investment projects A and B

Projects	Gains (RMB 10000 yuan)	Winning probabilities
Project A	1	0.1
	1,000,000	0.9
Project B	2	0.1
	3	0.9

Property 3. If we let $F(x)$, $G(x)$, and $L(x)$ denote the cumulative distribution functions of the evaluations of alternatives A, B, and C, respectively, then we have the following:

1) If $F(x)$ FSD $G(x)$ and $G(x)$ FSD $L(x)$, then $F(x)$ FSD $L(x)$, i.e., A FSD C;
2) If $F(x)$ SSD $G(x)$ and $G(x)$ SSD $L(x)$, then $F(x)$ SSD $L(x)$, i.e., A SSD C;
3) If $F(x)$ TSD $G(x)$ and $G(x)$ TSD $L(x)$, then $F(x)$ TSD $L(x)$, i.e., A TSD C;
4) If $F(x)$ AFSD $G(x)$ and $G(x)$ AFSD $L(x)$, then $F(x)$ AFSD $L(x)$, i.e., A AFSD C;
5) If $F(x)$ ASSD $G(x)$ and $G(x)$ ASSD $L(x)$, then $F(x)$ ASSD $L(x)$, i.e., A ASSD C; and.
6) If $F(x)$ ATSD $G(x)$ and $G(x)$ ATSD $L(x)$, then $F(x)$ ATSD $L(x)$, i.e., A ATSD C.

Proof First, we prove the conclusion 1. If $F(x)$ FSD $G(x)$ and $G(x)$ FSD $L(x)$, then we know $F(x) \le G(x)$ and $G(x) \le L(x)$ for any $x \in [a,b]$, $F(x) \ne G(x)$, and $G(x) \ne L(x)$. Thus, we have $F(x) \le L(x)$ (i.e., $F(x) - L(x) \le 0$) for any $x \in [a,b]$ and $F(x) \ne L(x)$. Therefore, we know $F(x)$ FSD $L(x)$ according to Definition 1. In a similar way, the conclusions 2 and 3 can also be proved.

Second, we prove conclusion 4. If $F(x)$ AFSD $G(x)$ and $G(x)$ AFSD $L(x)$, then we know

$$0 \le \int_{\Omega_1}[F(x)-G(x)]\,dx / \int_a^b |F(x)-G(x)|\,dx < 0.5, \tag{1}$$

$$0 \le \int_{\Omega_1'}[G(x)-L(x)]\,dx / \int_a^b |G(x)-L(x)|\,dx < 0.5, \tag{2}$$

where $\Omega_1 = \{x|\, G(x) < F(x),\ x \in [a,b]\}$ and $\Omega_1{}' = \{x|\, L(x) < G(x),\ x \in [a,b]\}$.

Eq. (1) can be written as follows:

$$\begin{aligned} & 0 \le 2\int_{\Omega_1}[F(x)-G(x)]\,dx < \int_a^b |F(x)-G(x)|\,dx \\ \Leftrightarrow\, & 0 \le 2\int_{\Omega_1}[F(x)-G(x)]\,dx < \int_{\Omega_1}[F(x)-G(x)]\,dx + \int_{\overline{\Omega}_1}[G(x)-F(x)]\,dx \\ \Leftrightarrow\, & 0 \le \int_{\Omega_1}[F(x)-G(x)]\,dx < \int_{\overline{\Omega}_1}[G(x)-F(x)]\,dx \end{aligned} \tag{3}$$

Eq. (2) can be written as follows:

$$\begin{aligned}&0\le 2\int_{\Omega_1'}[G(x)-L(x)]\,dx<\int_a^b|G(x)-L(x)|\,dx\\&\Leftrightarrow 0\le 2\int_{\Omega_1'}[G(x)-L(x)]\,dx<\int_{\Omega_1'}[G(x)-L(x)]\,dx+\int_{\overline{\Omega_1}'}[L(x)-G(x)]\,dx\\&\Leftrightarrow 0\le\int_{\Omega_1'}[G(x)-L(x)]\,dx<\int_{\overline{\Omega_1}'}[L(x)-G(x)]\,dx\end{aligned}\tag{4}$$

By adding Eq. (3) and Eq. (4), we can obtain the following:

$$\begin{aligned}&0\le\int_{\Omega_1}[F(x)-G(x)]\,dx+\int_{\Omega_1'}[G(x)-L(x)]\,dx<\int_{\overline{\Omega_1}}[G(x)-F(x)]\,dx+\int_{\overline{\Omega_1}'}[L(x)-G(x)]\,dx\\&\Leftrightarrow 0\le\int_{\Omega_1}[F(x)-G(x)]\,dx+\int_{\overline{\Omega_1}}[-G(x)+F(x)]\,dx<\int_{\overline{\Omega_1}'}[L(x)-G(x)]\,dx+\int_{\Omega_1'}[-G(x)+L(x)]\,dx\\&\Leftrightarrow 0\le\int_{[a,b]}[F(x)-G(x)]\,dx<\int_{[a,b]}[L(x)-G(x)]\,dx\Leftrightarrow 0<\int_{[a,b]}[L(x)-F(x)]\,dx\\&\Leftrightarrow 0<\int_{\overline{\Omega_1}''}[L(x)-F(x)]\,dx+\int_{\Omega_1''}[L(x)-F(x)]\,dx\\&\Leftrightarrow\int_{\Omega_1''}[F(x)-L(x)]\,dx<\int_{\overline{\Omega_1}''}[L(x)-F(x)]\,dx\\&\Leftrightarrow 2\int_{\Omega_1''}[F(x)-L(x)]\,dx<\int_{\overline{\Omega_1}''}[L(x)-F(x)]\,dx+\int_{\Omega_1''}[F(x)-L(x)]\,dx\\&\Leftrightarrow 2\int_{\Omega_1''}[F(x)-L(x)]\,dx<\int_{[a,b]}|F(x)-L(x)|\,dx\\&\Leftrightarrow\int_{\Omega_1''}[F(x)-L(x)]\,dx/\int_a^b|F(x)-L(x)|\,dx<0.5\end{aligned}\tag{5}$$

where $\Omega_1''=\{x|\,L(x)<F(x),\ x\in[a,b]\}$. If $\Omega_1''=\varnothing$, then we know $\int_{\Omega_1''}[F(x)-L(x)]\,dx=0$, and if $\Omega_1''\neq\varnothing$, then we know $\int_{\Omega_1''}[F(x)-L(x)]\,dx>0$. Therefore, Eq. (5) can be written as follows:

$$0\le\int_{\Omega_1''}[F(x)-L(x)]\,dx/\int_a^b|F(x)-L(x)|\,dx<0.5.$$

Therefore, according to Definition 4, we know that $F(x)$ AFSD $L(x)$, i.e., A AFSD C. In a similar way, conclusions 5 and 6 can also be proved.

The Proposed SMADM Method

In this section, a novel SMADM method based on the SD and ASD rules is proposed. All notations used in this paper are shown in Table 2.

Consider an SMADM problem. Let $A=\{A_i|\,i\in 1,2,\cdots,m\}$ be a set of m alternatives, where A_i denotes the ith alternative. In addition, let $C=\{C_j|\,j\in 1,2,\cdots,n\}$ be a set of n attributes, where C_j is the jth attribute and $C_1,C_2,\cdots,\ C_n$ are independent of each other. Let $w=(w_1,w_2,\ldots,w_n)^T$ be an attribute weight vector, where w_j is the weight or importance of attribute C_j,

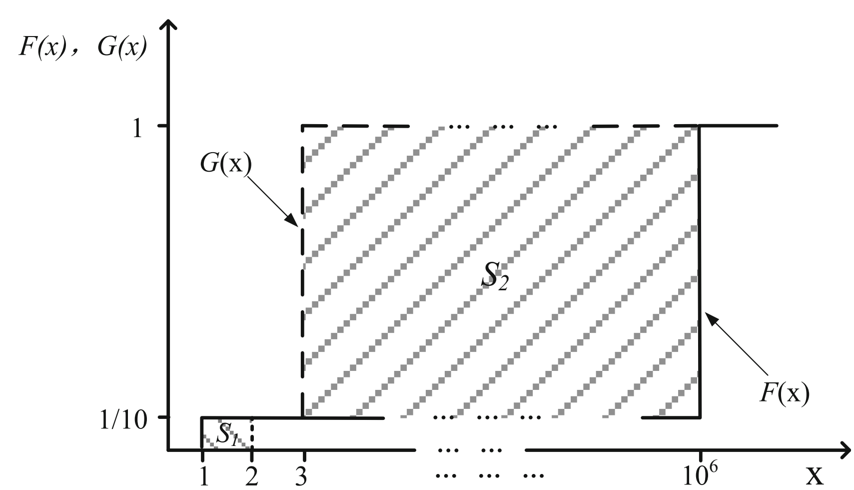

Fig. 1 The cumulative distribution functions of the gains of investment projects A and B

Table 2 Notations and explanations

Notations	Explanations
m	The total number of alternatives
n	The total number of attributes
$i \in \{1, 2, \cdots, m\}$	The index of alternatives
$j \in \{1, 2, \cdots, n\}$	The index of attributes
$A = \{A_1, A_2, \cdots, A_m\}$	A set of m alternatives, where A_i denotes the ith alternative
$C = \{C_1, C_2, \cdots, C_n\}$	A set of n attributes, where C_j denotes the jth attribute. We consider that all the attributes are independent with each other
$w = (w_1, w_2, \ldots, w_n)^T$	The vector of attribute weights, where w_j denotes the weight of the jth attribute, $\sum_{j=1}^{n} w_j = 1$, $w_j \geq 0$, $j = 1, 2, \cdots, n$
$X = [x_{ij}]_{m \times n}$	The decision matrix, where x_{ij} is the evaluation result of alternative A_i concerning attribute C_j
$f_{ij}(x)$	The probability density function of the stochastic variable x_{ij}
$x_{ij}^1, x_{ij}^2, \ldots, x_{ij}^q$	The possible values of the stochastic variable x_{ij}.
p_{ij}^h	The probability of the discrete stochastic variable $x_{ij} = x_{ij}^h$
$F_{ij}(x)$	The cumulative distribution function of x_{ij}
u_{ij}	The mathematical expectation of x_{ij}
$\tilde{R}_j = \left[\tilde{r}_{ik}^j\right]_{m \times m}$	The dominance relation matrix, where $\tilde{r}_{ik}^j$ denotes the SD or ASD relation between alternatives A_i and A_k concerning attribute C_j
$\overline{\text{SD}}$	The SD or ASD relation between a pair of alternatives, $\overline{\text{SD}} \in \{\text{FSD, SSD, TSD, AFSD, ASSD, ATSD}\}$
p_j	The DM's preference threshold concerning attribute C_j.
$P_j(A_i, A_k)$	The priority degree that alternative A_i is superior to alternative $A_k$concerning attribute C_j
$D = [\sigma_{ik}]_{m \times m}$	The overall dominance priority matrix, where σ_{ik} denotes the overall priority degree that alternative A_i is superior to alternative A_k
σ_k^+	The maximum value in the kth column of matrix D
σ_k^-	The maximum value in the kth row of matrix D
$X^+ = \left(\sigma_1^+, \sigma_2^+, \cdots, \sigma_m^+\right)$	The vector of column maximum values
$X^- = \left(\sigma_1^-, \sigma_2^-, \cdots, \sigma_m^-\right)^T$	The vector of row maximum values
S_i^+, S_i^-	Deviation degrees
I_i	The closeness coefficient of alternative A_i

such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \geq 0, j = 1, 2, \cdots, n$. Let $X = [x_{ij}]_{m \times n}$ be a decision matrix, where x_{ij} is the evaluation result of alternative A_i concerning attribute C_j, $i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$. In this paper, we consider that x_{ij} is a stochastic variable with the cumulative distribution function $F_{ij}(x)$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$. Let u_{ij} denote the mathematic expectation of x_{ij}. According to probability theory, if x_{ij} is a continuous stochastic variable, then $u_{ij} = \int_{-\infty}^{\infty} x f_{ij}(x) dx$, where $f_{ij}(x)$ denotes the probability density function; and if x_{ij} is a discrete stochastic variable, then $u_{ij} = \sum_{h=1}^{q} x_{ij}^h p_{ij}^h$ where p_{ij}^h denotes the probability of $x_{ij} = x_{ij}^h$, and the possible values of the stochastic variable x_{ij} are $x_{ij}^1, x_{ij}^2, \ldots, x_{ij}^q$. The problem that is addressed in this paper is how to rank alternatives or to select the most desirable alternative(s) based on the decision matrix X and the attribute weight vector w.

To solve the above problem, the SMADM method based on the SD and ASD rules is proposed. In the method, based on the SD and ASD rules, a procedure for identifying the dominance relation between each pair of alternatives is given. Then, according to the identified dominance relations, a priority function with respect to each attribute is constructed to measure the priority degrees of the pairwise comparisons of the alternatives. Further, using the simple weighted method, an overall priority degree for each pair of alternatives is

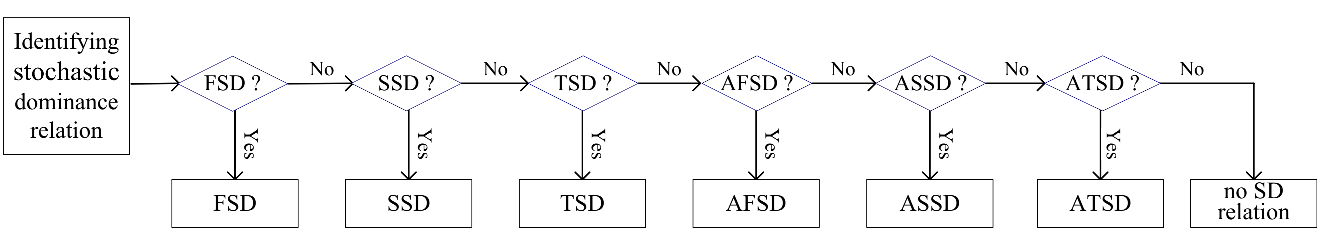

Fig. 2 The procedure for identifying the dominance relation between each pair of alternatives

Table 3 Details of the online ratings of the five candidate passenger cars concerning the four attributes

Candidate passenger cars	Rating scores	Attributes			
		C_1	C_2	C_3	C_4
A_1	1	0	0	1	0
	2	2	0	1	1
	3	4	2	3	5
	4	30	21	29	15
	5	34	47	36	49
A_2	1	0	0	1	1
	2	1	1	1	0
	3	11	4	4	10
	4	37	16	17	27
	5	24	52	50	35
A_3	1	0	0	1	2
	2	0	0	0	0
	3	2	0	4	3
	4	25	5	35	18
	5	63	85	50	67
A_4	1	0	1	1	2
	2	1	0	1	5
	3	1	3	9	17
	4	33	16	31	24
	5	40	55	33	27
A_5	1	0	0	0	0
	2	1	0	2	2
	3	1	3	6	5
	4	17	17	16	16
	5	22	21	17	18

Table 4 Probability distributions of the online ratings about the five candidate passenger cars concerning the four attributes

Candidate passenger cars	Rating scores	Attributes			
		C_1	C_2	C_3	C_4
A_1	1	0	0	1/70	0
	2	2/70	0	1/70	1/70
	3	4/70	2/70	3/70	5/70
	4	30/70	21/70	29/70	15/70
	5	34/70	47/70	36/70	49/70
A_2	1	0	0	1/73	1/73
	2	1/73	1/73	1/73	0
	3	11/73	4/73	4/73	10/73
	4	37/73	16/73	17/73	27/73
	5	24/73	52/73	50/73	35/73
A_3	1	0	0	1/90	2/90
	2	0	0	0	0
	3	2/90	0	4/90	3/90
	4	25/90	5/90	35/90	18/90
	5	63/90	85/90	50/90	67/90
A_4	1	0	1/75	1/75	2/75
	2	1/75	0	1/75	5/75
	3	1/75	3/75	9/75	17/75
	4	33/75	16/75	31/75	24/75
	5	40/75	55/75	33/75	27/75
A_5	1	0	0	0	0
	2	1/41	0	2/41	2/41
	3	1/41	3/41	6/41	5/41
	4	17/41	17/41	16/41	16/41
	5	22/41	21/41	17/41	18/41

calculated, and a priority degree matrix for the pairwise comparisons of alternatives is built. Finally, according to the priority degree matrix, an approach for ranking the alternatives is given.

To identify whether the SD relation or the ASD relation exists concerning each attribute between each pair of alternatives, the SD rules (i.e., Definitions 1–3) and the ASD rules (i.e., Definitions 4–6) are used. The procedure for identifying the SD or ASD relation between each pair of alternatives is shown in Fig. 2. In Fig. 2, the SD or ASD relation between each pair of alternatives concerning each attribute is first identified according to Definitions 1. If the dominance relation exists, then it belongs to the FSD relation; otherwise, Definition 2 is used to identify whether the SSD relation exists or not, and so on.

Using the procedure shown in Fig. 2, a dominance relation matrix $\tilde{R}_j = \left[\tilde{r}_{ik}^j\right]_{m\times m}$ with respect to each attribute can be constructed, where $\tilde{r}_{ik}^j$ denotes the SD or ASD relation between alternatives A_i and A_k concerning attribute C_j. The $\tilde{r}_{ik}^j$ can be expressed by

$$\tilde{r}_{ik}^j = \begin{cases} \text{FSD}, & F_{ij}(x) \text{ FSD } F_{kj}(x) \Leftrightarrow A_i \text{ FSD } A_k \\ \text{SSD}, & F_{ij}(x) \text{ SSD } F_{kj}(x) \Leftrightarrow A_i \text{ SSD } A_k \\ \text{TSD}, & F_{ij}(x) \text{ TSD } F_{kj}(x) \Leftrightarrow A_i \text{ TSD } A_k \\ \text{AFSD}, & F_{ij}(x) \text{ AFSD } F_{kj}(x) \Leftrightarrow A_i \text{ AFSD } A_k \\ \text{ASSD}, & F_{ij}(x) \text{ ASSD } F_{kj}(x) \Leftrightarrow A_i \text{ ASSD } A_k \\ \text{ATSD}, & F_{ij}(x) \text{ ATSD } F_{kj}(x) \Leftrightarrow A_i \text{ ATSD } A_k \\ -, & \text{no SD relation} \end{cases}$$

For the convenience of the analysis, let $\overline{\text{SD}}$ denote that the SD or ASD relation exists between a pair of alternatives,

Table 5 The dominance relation matrix concerning attribute C_1

	A_1	A_2	A_3	A_4	A_5
A_1	–	AFSD	–	–	–
A_2	–	–	–	–	–
A_3	FSD	FSD	–	FSD	FSD
A_4	FSD	FSD	–	–	SSD
A_5	FSD	AFSD	–	–	–

Table 6 The dominance relation matrix concerning attribute C_2

	A_1	A_2	A_3	A_4	A_5
A_1	–	SSD	–	–	FSD
A_2	–	–	–	–	AFSD
A_3	FSD	FSD	–	FSD	FSD
A_4	AFSD	AFSD	–	–	AFSD
A_5	–	–	–	–	–

Table 8 The dominance relation matrix concerning attribute C_4

	A_1	A_2	A_3	A_4	A_5
A_1	–	SSD	–	FSD	FSD
A_2	–	–	–	FSD	AFSD
A_3	AFSD	AFSD	–	FSD	AFSD
A_4	–	–	–	–	–
A_5	–	–	–	FSD	–

where $\overline{\text{SD}} \in \{\text{FSD, SSD, TSD, AFSD, ASSD, ATSD}\}$. Let u_{ij} and u_{kj} denote the expected values of the evaluations of alternatives A_i and A_k concerning attribute C_j, respectively. Let p_j denote the DM's preference threshold concerning attribute C_j [24, 26, 27]. Here, there are three possible relations with respect to attribute C_j between alternatives A_i and A_k:

1) Alternative A_i is strictly superior to alternative A_k if $F_{ij}(x)$ $\overline{\text{SD}}$ $F_{kj}(x)$ and $u_{ij} \geq u_{kj} + p_j$;
2) Alternative A_i is weaker superior to alternative A_k if $F_{ij}(x)$ $\overline{\text{SD}}$ $F_{kj}(x)$ and $u_{kj} < u_{ij} < u_{kj} + p_j$; and.
3) Alternative A_i is no different from alternative A_k if there is no $F_{ij}(x)$ $\overline{\text{SD}}$ $F_{kj}(x)$ or $F_{kj}(x)$ $\overline{\text{SD}}$ $F_{ij}(x)$.

Further, the priority degree that alternative A_i is superior to alternative A_k concerning attribute C_j, $P_j(A_i, A_k)$, can be calculated by

$$P_j(A_i, A_k) = \begin{cases} 1, & F_{ij}(x)\ \overline{\text{SD}}\ F_{kj}(x) \text{ and } u_{ij} \geq u_{kj} + p_j \\ \frac{u_{ij} - u_{kj}}{p_j}, & F_{ij}(x)\ \overline{\text{SD}}\ F_{kj}(x) \text{ and } u_{kj} < u_{ij} < u_{kj} + p_j \\ 0, & \text{else} \end{cases} \tag{6}$$

Obviously, $P_j(A_i, A_k) \in [0, \ 1]$. The greater the value of $P_j(A_i, A_k)$ is, the greater the degree that alternative A_i is superior to alternative A_k will be.

Then, using the simple weighted method, the overall priority degree that alternative A_i is superior to alternative A_k, σ_{ik}, can be calculated by

$$\sigma_{ik} = \sum_{j=1}^{n} w_j p_j(A_i, A_k), \cdot i, k = 1, 2, \cdots, m, \cdot i \neq k \tag{7}$$

Table 7 The dominance relation matrix concerning attribute C_3

	A_1	A_2	A_3	A_4	A_5
A_1	–	–	–	AFSD	AFSD
A_2	AFSD	–	AFSD	AFSD	AFSD
A_3	FSD	–	–	FSD	AFSD
A_4	–	–	–	–	AFSD
A_5	–	–	–	–	–

Here, the overall priority degree σ_{ik} can be regarded as the confidence that alternative A_i is superior to alternative A_k, where $0 \leq \sigma_{ik} \leq 1$. The larger the value of σ_{ik} is, the greater the confidence will be. Based on σ_{ik}, we can establish the overall priority degree matrix of the pairwise comparisons of alternatives as

$$D = [\sigma_{ik}]_{m \times m} = \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} \overset{\begin{matrix} A_1 & A_2 & \cdots & A_m \end{matrix}}{\begin{pmatrix} - & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & - & \cdots & \sigma_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & - \end{pmatrix}}$$

According to the overall dominance priority matrix D, an approach is given to determine the ranking of alternatives. The detailed description of the approach is given below.

Based on matrix D, two vectors $X^+ = (\sigma_1^+, \sigma_2^+, \cdots, \sigma_m^+)$ and $X^- = (\sigma_1^-, \sigma_2^-, \cdots, \sigma_m^-)^T$ can be constructed, respectively, where σ_k^+ and σ_k^- are respectively given by

$$\sigma_k^+ = \max\{\sigma_{ik} | i = 1, \ 2, \ \cdots, m; \ i \neq k\}, \qquad k = 1, 2, \cdots, m \tag{8}$$

$$\sigma_k^- = \max\{\sigma_{ki} | i = 1, \ 2, \ \cdots, m; \ i \neq k\}, \qquad k = 1, 2, \cdots, m \tag{9}$$

In Eq. (8), σ_k^+ is the maximum value in the kth column of matrix D. σ_k^+ denotes the greatest overall degree that an alternative among the other $m-1$ alternatives is superior to alternative A_k. In Eq. (9), σ_k^- is the maximum value in the kth row of matrix D. σ_k^- denotes the greatest overall degree that alternative A_k is superior to an alternative among the other $m-1$ alternatives.

According to the matrix D and vectors X^+ and X^-, two deviation degrees S_i^+ and S_i^- can be respectively calculated as

Table 9 The overall dominance degree matrix

	A_1	A_2	A_3	A_4	A_5
A_1	–	0.139	0.000	0.180	0.175
A_2	0.049	–	0.025	0.169	0.162
A_3	0.185	0.300	–	0.320	0.328
A_4	0.044	0.123	0.000	–	0.067
A_5	0.032	0.109	0.000	0.060	–

Table 10 The evaluations provided by the experts

Attribute	Rating scores	Alternatives									
		A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
C_1	1							1/7	1/7		
	2	3/7	1/7						2/7		1/7
	3	1/7				1/7			2/7		2/7
	4		2/7						1/7		2/7
	5	2/7	1/7	3/7	1/7			3/7	1/7	2/7	1/7
	6		2/7	1/7		2/7	1/7	1/7		1/7	
	7	1/7		1/7	1/7	2/7	2/7			3/7	1/7
	8		1/7	2/7	1/7		2/7	1/7		1/7	
	9				3/7	2/7					
	10				1/7		2/7	1/7			
C_2	1							1/7	3/7		
	2	2/7						3/7	3/7		1/7
	3	1/7			1/7		4/7	1/7		1/7	
	4				1/7				1/7	1/7	
	5	2/7				1/7		1/7			
	6		1/7	1/7	1/7	2/7		1/7		1/7	
	7		1/7			1/7	1/7			4/7	2/7
	8	1/7	3/7	2/7	3/7	2/7	2/7				3/7
	9	1/7	2/7	3/7	1/7	1/7					
	10			1/7							1/7
C_3	1								2/7		1/7
	2							3/7	1/7		2/7
	3	1/7			1/7			1/7	4/7	1/7	
	4	3/7					1/7	1/7		2/7	
	5		1/7				1/7	2/7		2/7	
	6	1/7									2/7
	7		1/7		1/7	2/7				2/7	2/7
	8	1/7	2/7	4/7	2/7	3/7	2/7				
	9	1/7	3/7	3/7	1/7	1/7	1/7				
	10				2/7	1/7	2/7				
C_4	1								2/7		
	2									1/7	
	3	3/7						1/7			
	4								1/7	1/7	
	5	2/7					1/7	1/7	2/7		
	6					1/7	1/7		1/7	3/7	3/7
	7			1/7		1/7	1/7				1/7
	8	1/7	4/7	4/7	3/7	3/7	2/7	3/7	1/7	1/7	1/7
	9		2/7		1/7	1/7	1/7	1/7			1/7
	10	1/7	1/7	2/7	3/7	1/7	1/7	1/7		1/7	1/7

$$S_i^+ = \sum_{\substack{k=1 \\ k \neq i}}^{m} \left| \sigma_{ik} - \sigma_k^+ \right| = \sum_{\substack{k=1 \\ k \neq i}}^{m} \sigma_k^+ - \sum_{\substack{k=1 \\ k \neq i}}^{m} \sigma_{ik}, \quad i = 1, 2, \cdots, m \tag{10}$$

$$S_i^- = \sum_{\substack{k=1 \\ k \neq i}}^{m} \left| \sigma_{ki} - \sigma_k^- \right| = \sum_{\substack{k=1 \\ k \neq i}}^{m} \sigma_k^- - \sum_{\substack{k=1 \\ k \neq i}}^{m} \sigma_{ki}, \quad i = 1, 2, \cdots, m \tag{11}$$

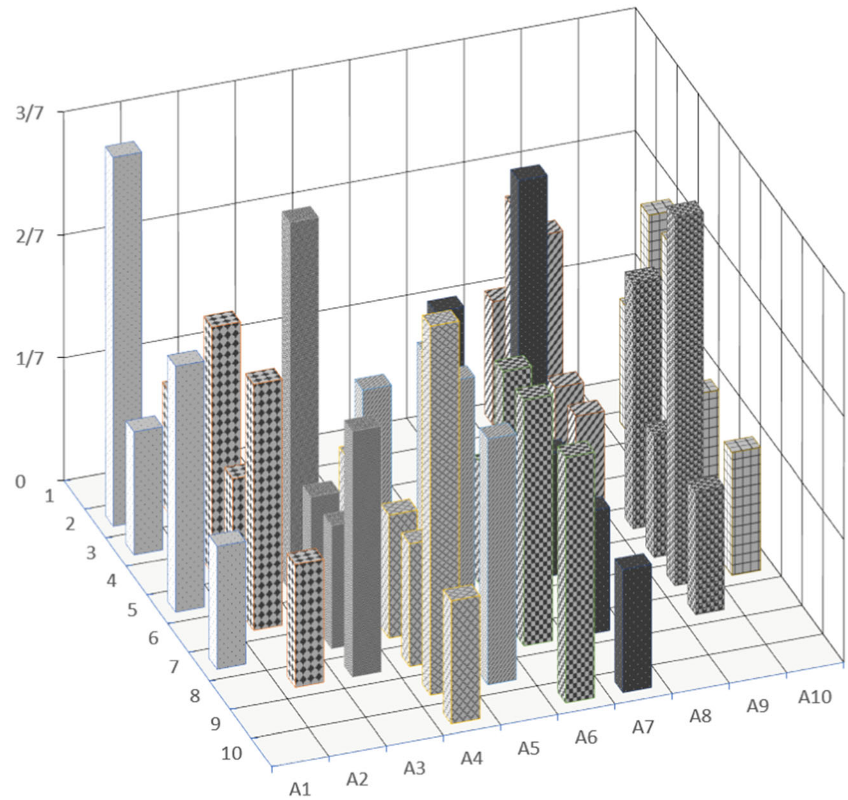

Fig. 3 The graphical representation of the distributions concerning attribute C_1

In Eq. (10), $\sum_{k=1,k\neq i}^{m}\sigma_{ik}$ denotes the total confidence that alternative A_i is superior to the other alternatives. The greater $\sum_{k=1,k\neq i}^{m}\sigma_{ik}$ is, the smaller the deviation S_i^+ will be, and the better alternative A_i will be. In Eq. (11), $\sum_{k=1,k\neq i}^{m}\sigma_{ki}$ denotes the total confidence that the other alternatives are superior to alternative A_i. The smaller $\sum_{k=1,k\neq i}^{m}\sigma_{ki}$ is, the greater the deviation S_i^- will be, and the better alternative A_i will be.

According to the obtained S_i^+ and S_i^-, the closeness coefficient I_i of alternative A_i can be calculated as

$$I_i = \frac{S_i^-}{S_i^+ + S_i^-}, \qquad i = 1, 2, \cdots, m \tag{12}$$

Obviously, the greater the value of I_i is, the better alternative A_i will be. Therefore, according to the values of $I_1, I_2, \ldots, I_m$, the ranking of alternatives can be determined.

In summary, the calculation steps of the SMADM method based on the SD and ASD rules are given as follows.

Step 1. According to the SD and ASD rules, i.e., Definitions 1–6, identify the SD or ASD relation concerning each attribute between each pair of alternatives, and establish the dominance relation matrices, i.e., $\tilde{R}_j = \left[\tilde{r}_{ik}^j\right]_{m\times m}, j = 1, 2, \cdots, n$.

Table 11 The dominance relation matrix concerning attribute C_1.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	–	–	–	–	–	–	FSD	–	–
A_2	FSD	–	–	–	–	–	–	FSD	–	FSD
A_3	FSD	FSD	–	–	–	–	SSD	FSD	–	FSD
A_4	FSD	FSD	FSD	–	FSD	AFSD	FSD	FSD	FSD	FSD
A_5	FSD	FSD	AFSD	–	–	–	SSD	FSD	AFSD	FSD
A_6	FSD	FSD	FSD	–	FSD	–	FSD	FSD	FSD	FSD
A_7	AFSD	AFSD	–	–	–	–	–	FSD	–	AFSD
A_8	–	–	–	–	–	–	–	–	–	–
A_9	FSD	FSD	FSD	–	–	–	SSD	FSD	–	FSD
A_{10}	SSD	–	–	–	–	–	–	FSD	–	–

Step 2. Calculate the overall priority degree of each pair of alternatives using Eqs. (6) and (7) and construct the overall priority degree matrix $D = [\sigma_{ik}]_{m \times m}$.

Step 3. Construct the vectors X^{+} and X^{-} using Eqs. (8) and (9).

Step 4. Calculate the deviations S_i^{+} and S_i^{-} using Eqs. (10) and (11).

Step 5. Calculate the closeness coefficient I_i using Eq. (12) and determine the ranking of the alternatives.

An Application to Support Online Purchasing Decision

In recent years, online shopping has become a universal phenomenon in China [39]. Online product ratings, as a type of electronic word-of-mouth, play an important role in helping consumers select desirable products, but it is difficult for consumers to read a large number of online ratings on an e-commerce website. To support consumers' purchase decisions, how to rank the candidate products based on online product ratings and consumers' preferences is a noteworthy problem.

In this section, the problem of passenger car selection based on online product ratings is considered. A consumer wants to purchase a passenger car. The five candidate passenger cars assessed by the consumer are the following: Passat (A_1), Mondeo (A_2), Atenza (A_3), Lamando (A_4), and Sonata (A_5). The four attributes of the passenger cars assessed by the consumer are the following: power (C_1), control (C_2), comfortability (C_3), and cost performance (C_4), and the attribute weight vector provided by the consumer is $w = (0.35, 0.15, 0.3, 0.2)^T$. Here, the preference threshold for each attribute is $p_j = 1$ ($j = 1, 2, 3, 4$). To select a desirable passenger car, the corresponding online product ratings are extracted from the website of *Autohome* (http://www.autohome.com.cn/). The numbers of online ratings of the five candidate passenger cars are 70, 73, 90, 75, and 41, respectively. The details of the online ratings of the five candidate passenger cars concerning the four attributes are shown in Table 3. The probability distributions of the online ratings about the five candidate passenger cars concerning the four attributes are shown in Table 4. The proposed method is used to rank the five candidate passenger cars, and the computation procedure is briefly described as follows.

First, according to the SD rules and the ASD rules, i.e., Definitions 1–6, the SD or ASD relation between each pair of candidate passenger cars is identified, and the dominance relation matrices concerning the four attributes (i.e., $\tilde{R}_1 = [\tilde{r}_{ik}^1]_{m\times m}$, $\tilde{R}_2 = [\tilde{r}_{ik}^2]_{m\times m}$, $\tilde{R}_3 = [\tilde{r}_{ik}^3]_{m\times m}$, and $\tilde{R}_4 = [\tilde{r}_{ik}^4]_{m\times m}$) are established, which are shown in Tables 5, 6, 7 and 8, respectively. Then, using Eqs. (6) and (7), the overall priority degree for each pair of candidate passenger cars is calculated, and the overall priority degree matrix $D = [\sigma_{ik}]_{m \times m}$ is constructed, which is shown in Table 9. According to Table 9, using Eqs. (8) and (9), two vectors X^{+} and X^{-} can be constructed as $X^{+} = (0.185, 0.300, 0.025, 0.320, 0.328)$ and $X^{-} = (0.180, 0.169, 0.328, 0.123, 0.109)^T$. Using Eqs. (10) and (11), the deviations S_i^{+} and S_i^{-} can be calculated

Table 12 The dominance relation matrix concerning attribute C_2

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	–	–	–	–	–	FSD	FSD	–	–
A_2	FSD	–	–	FSD	FSD	FSD	FSD	FSD	FSD	SSD
A_3	FSD	FSD	–	FSD	FSD	FSD	FSD	FSD	FSD	FSD
A_4	FSD	–	–	–	–	FSD	FSD	FSD	FSD	–
A_5	FSD	–	–	SSD	–	FSD	FSD	FSD	FSD	–
A_6	SSD	–	–	–	–	–	FSD	FSD	–	–
A_7	–	–	–	–	–	–	–	FSD	–	–
A_8	–	–	–	–	–	–	–	–	–	–
A_9	SSD	–	–	–	–	SSD	FSD	FSD	–	–
A_{10}	FSD	–	–	AFSD	AFSD	AFSD	FSD	FSD	AFSD	–

Table 13 The dominance relation matrix concerning attribute C_3

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	–	–	–	–	–	FSD	FSD	AFSD	SSD
A_2	FSD	–	–	SSD	–	SSD	FSD	FSD	FSD	FSD
A_3	FSD	FSD	–	SSD	SSD	SSD	FSD	FSD	FSD	FSD
A_4	FSD	–	–	–	–	AFSD	FSD	FSD	FSD	FSD
A_5	FSD	SSD	–	SSD	–	SSD	FSD	FSD	FSD	FSD
A_6	FSD	–	–	–	–	–	FSD	FSD	FSD	FSD
A_7	–	–	–	–	–	–	–	FSD	–	–
A_8	–	–	–	–	–	–	–	–	–	–
A_9	–	–	–	–	–	–	FSD	FSD	–	SSD
A_{10}	–	–	–	–	–	–	AFSD	FSD	–	–

Table 14 The dominance relation matrix concerning attribute C_4

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	–	–	–	–	–	–	SSD	–	–
A_2	FSD	–	SSD	–	FSD	FSD	FSD	FSD	FSD	FSD
A_3	FSD	–	–	–	FSD	FSD	FSD	FSD	FSD	FSD
A_4	FSD	FSD	FSD	–	FSD	FSD	FSD	FSD	FSD	FSD
A_5	FSD	–	–	–	–	FSD	FSD	FSD	FSD	FSD
A_6	FSD	–	–	–	–	–	SSD	FSD	FSD	–
A_7	FSD	–	–	–	–	–	–	FSD	FSD	–
A_8	–	–	–	–	–	–	–	–	–	–
A_9	AFSD	–	–	–	–	–	–	FSD	–	–
A_{10}	FSD	–	–	–	–	AFSD	SSD	FSD	FSD	–

as S_1^+ =0.479, S_2^+ =0.453, S_3^+ =0, S_4^+ =0.604, and S_5^+ =0.629; and S_1^- =0.419, S_2^- =0.069, S_3^- =0.556, S_4^- =0.057, and S_5^- =0.068. Finally, using Eq. (12), the closeness coefficient of each candidate passenger car can be calculated as I_1=0.467, I_2=0.132, I_3=1, I_4=0.086, and I_5=0.098. Thus, according to the closeness coefficients, the ranking of the five candidate passenger cars can be determined as $A_3 \succ A_1 \succ A_2 \succ A_5 \succ A_4$. The ranking result can be used to support the consumer in selecting the most desirable passenger car.

Comparative Analysis

In this section, an example that was investigated by Zaras and Martel [33] and Nowak [24] is used to compare the proposed method with the existing SMADM methods. Consider the problem of selecting the most desirable computer development project(s) from ten alternative projects $A_1, A_2, \cdots, A_{10}$. The considered attributes include the following: personal resources efforts (C_1), discounted profits (C_2), chances of success (C_3), and technological orientation (C_4). The attribute weight vector provided by the DM is $w = (0.09, 0.55, 0.27, 0.09)^T$ and the preference threshold for each attribute is $p_j = 1$ ($j = 1, 2, 3, 4$). To solve the problem, the seven experts are invited to participate in the decision analysis. Here, the seven experts provide evaluations of the projects (or alternatives) with respect to the attributes on a scoring scale of ten (1: the worst and 10: the best), as shown in Table 10, and we can see that experts' evaluations follow the form of probability distributions. According to the data in Table 10, the graphical representation of the distributions concerning attribute C_1, as an example, is given in Fig. 3.

To solve the project selection problem, the method proposed in this paper is used and the procedure is summarized as follows.

First, according to the SD rules and the ASD rules, i.e., Definitions 1–6, the SD or ASD relation between each pair of alternatives is identified. The dominance relation matrices concerning the four attributes (i.e., $\tilde{R}_1 = [\tilde{r}_{ik}^1]_{m\times m}$, $\tilde{R}_2 = [\tilde{r}_{ik}^2]_{m\times m}$, $\tilde{R}_3 = [\tilde{r}_{ik}^3]_{m\times m}$ and $\tilde{R}_4 = [\tilde{r}_{ik}^4]_{m\times m}$) are established, which are shown in Tables 11, 12, 13 and 14, respectively. Then, using Eqs. (6) and (7), the overall priority degree of each pair of alternatives is calculated and the overall priority degree matrix $D = [\sigma_{ik}]_{m\times m}$ is constructed, which is shown in Table 15. According to Table 15, two vectors X^+ and X^- can be constructed using Eqs. (8) and (9) as X^+=(1, 0.557, 0.09, 0.704, 0.667, 0.822, 1, 1, 0.935, 1) and X^-=(0.987, 1, 1, 1, 1, 1, 1, 0, 1, 1). Using Eqs. (10) and (11), the deviations S_i^+ and S_i^- can be calculated as S_1^+ =4.583, S_2^+ =0.848, S_3^+ =0.063, S_4^+ =1.818, S_5^+ =1.568, S_6^+ =3.398, S_7^+ =5.531, S_8^+ =6.775, S_9^+ =3.495, and S_{10}^+ =3.425; and S_1^- =2.167, S_2^- =6.930, S_3^- =7.756, S_4^- =6.108, S_5^- =6.590, S_6^- =4.064, S_7^- =0.668, S_8^- =0.000, S_9^- =3.825, and S_{10}^- =4.303. Finally, using Eq. (12), the closeness coefficient of each alternative can be calculated as I_1=0.321, I_2=0.891, I_3=0.992, I_4=0.771, I_5=0.808, I_6=0.545, I_7=0.108, I_8=0, I_9=0.523, and I_{10}=0.557. Thus, according to the closeness coefficients, the ranking of the alternatives can be determined as $A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_{10} \succ A_6 \succ A_9 \succ A_1 \succ A_7 \succ A_8$.

It is necessary to point out that the ranking result obtained by [33] is $A_3, A_4 \succ A_2, A_5 \succ A_6, A_9, A_{10} \succ A_1, A_7 \succ A_8$, and the ranking result obtained by [24] is $A_3 \succ A_2 \succ A_4, A_5 \succ A_6 \succ A_9, A_{10} \succ A_1 \succ A_7 \succ A_8$. Obviously, it can be seen from these two ranking results that the ranking positions of A_3 and A_4; A_2 and A_5; A_6, A_9, and A_{10}; and A_1 and A_7 cannot be distinguished using the method proposed by [33]; and the ranking positions of A_4 and A_5; and A_9 and A_{10} cannot be distinguished using the method proposed by [24]. Using the method proposed in this paper, the ranking result of alternatives is $A_3 \succ A_2 \succ A_5 \succ A_4 \succ A_{10} \succ A_6 \succ A_9 \succ A_1 \succ A_7 \succ A_8$. This means that a more precise ranking of alternatives can be obtained. Thus, it can be concluded that ASD rules are important supplements of the SD rules for identifying dominance relations. The obtained ASD relations are valuable decision information for determining more precise ranking result of alternatives.

Table 15 The overall priority degree matrix

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	0	0	0	0	0	0.82	0.987	0.116	0.270
A_2	1	–	0	0.55	0.473	0.681	0.91	1	0.91	0.846
A_3	1	0.557	–	0.704	0.667	0.822	0.962	1	0.91	1
A_4	1	0.09	0.09	–	0.09	0.693	1	1	0.841	0.450
A_5	1	0.166	0.038	0.312	–	0.705	0.974	1	0.935	0.411
A_6	0.527	0.09	0.09	0	0.09	–	0.935	1	0.45	0.373
A_7	0.09	0.064	0	0	0	0	–	1	0	0.09
A_8	0	0	0	0	0	0	0	–	0	0
A_9	0.64	0.09	0.013	0	0	0.473	0.885	1	–	0.244
A_{10}	0.576	0	0	0.314	0.077	0.55	0.833	1	0	–

Conclusions

This paper proposes a method for SMADM based on the SD and ASD rules. Using the SD and ASD rules, pairwise comparisons of alternatives concerning each attribute can be conducted, and the dominance relation matrix of the pairwise comparisons of alternatives with regard to each attribute can be built. An approach to rank alternatives is given. Compared with the existing studies, the proposed method has distinct characteristics as discussed below.

First, both the SD rules and the ASD rules are used in decision analysis. This is a new attempt to solve the SMADM problem.

Second, in the proposed method, both the SD rules and the ASD rules are used to identify the SD or ASD relation between each pair of alternatives. It should be noted that the identified ASD relations are important supplements of SD relations. This makes it possible to compare alternatives more clearly. Nevertheless, using the existing SMADM methods based on the SD rules, the dominance relations between some pairs of alternatives cannot be identified, which would lead to the lack of a clear ranking result of alternatives.

It is important to highlight that since the proposed method is new and different from the existing SMADM methods, it can give the DM one more choice for solving the practical SMADM problem. In addition to supplementing the existing SMADM methods, the proposed method is also important for developing theories and methods for SMADM.

The study also has some limitations, which may serve as avenues for future research. First, even if the ASD rules are used, there may still be some missing dominance relations between some pairs of alternatives. Thus, it is necessary to develop some new SD rules for comparing stochastic variables. In addition, to support DMs to facilitate the use of the method proposed in this paper, the decision support system needs to be developed. Furthermore, SMADM methods based on artificial intelligence methods should be given more attention.

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Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

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