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# Bayesian inverse kinematics vs. least-squares inverse kinematics in estimates of planar postures and rotations in the absence of soft tissue artifact

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ABSTRACT

A variety of inverse kinematics (IK) algorithms exist for estimating postures and displacements from a set of noisy marker positions, typically aiming to minimize IK errors by distributing errors amongst all markers in a least-squares (LS) sense. This paper describes how Bayesian inference can contrastingly be used to maximize the probability that a given stochastic kinematic model would produce the observed marker positions. We developed Bayesian IK for two planar IK applications: (1) kinematic chain posture estimates using an explicit forward kinematics model, and (2) rigid body rotation estimates using implicit kinematic modeling through marker displacements. We then tested and compared Bayesian IK results to LS results in Monte Carlo simulations in which random marker error was introduced using Gaussian noise amplitudes ranging uniformly between 0.2 mm and 2.0 mm. Results showed that Bayesian IK was more accurate than LS-IK in over 92% of simulations, with the exception of one center-of-rotation coordinate planar rotation, for which Bayesian IK was more accurate in only 68% of simulations. Moreover, while LS errors increased with marker noise, Bayesian errors were comparatively unaffected by noise amplitude. Nevertheless, whereas the LS solutions required average computational durations of less than 0.5 s, average Bayesian IK durations ranged from 11.6 s for planar rotation to over 2000 s for kinematic chain postures. These results suggest that Bayesian IK can yield order-of-magnitude IK improvements for simple planar IK, but also that its computational demands may make it impractical for some applications.



## 1. Introduction

Marker-based inverse kinematics (IK) is a fundamental component of motion capture, and a variety of IK algorithms exist in the literature for estimating segmental pose from noisy marker measurement (Aristidou and Lasenby, 2009). In Biomechanics, a wide variety of IK algorithms exist for minimizing noisy marker errors in a least-squares (LS) sense (Söderkvist and Wedin, 1993; Halvorsen et al., 1999; Gamage and Lasenby, 2002; Halvorsen, 2003; Schwartz and Rozumalski, 2005; Wells et al., 2017), including near-real time LS algorithms (Borbély et al., 2017; Pizzolato et al., 2016). Augmented LS algorithms have also been proposed, including for example Kalman filtering, which can substantially reduce LS errors (De Groote et al., 2008). As far as we are aware, all commonly employed biomechanics-specific software packages including Visual3D (C-Motion, Inc., Germantown, USA) and OpenSim (Delp et al., 2007) use LS-based IK. Nevertheless, in the broader literature, and in the robotics literature in particular, a variety of non-LS IK algorithms have been proposed, including artificial neural networks (Duka, 2014), heuristic methods (Aristidou and Lasenby, 2011), and Bayesian inference (Courty and Arnaud, 2008).

This paper focusses on Bayesian IK because it is known that Bayesian algorithms usually yield more accurate results than LS algorithms for generic (non-IK) parameter estimation problems, even for relatively simple cases like regression (Elster and Wübbeler, 2017) and multivariate mean estimation (Yuan et al., 2016). Moreover, it has been shown in the animation literature that optimized Bayesian implementations can outperform LS methods in animation IK problems involving a large number of

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degrees of freedom (more than 30) (Courty and Arnaud, 2008). Nevertheless, no systematic comparison of Bayesian vs. IK has previously been performed for simple kinematic models, so the minimal, general kinematic characteristics necessary for Bayesian IK and LS-IK diverge are presently unknown.

In generic (non-IK) applications, both LS and Bayesian approaches can be interpreted as follows. Given a set of observed values $y$, construct a forward model $f(q) = \hat{y}$, where $q$ are arbitrary parameters to be estimated, and $\hat{y}$ are the predicted values. Both LS and Bayesian approaches aim to estimate $q$ given $y$ and the forward model. If the observed data $y$ have zero error, and if the forward model is perfect, then there is a unique solution for $q$ and the LS and Bayesian approaches are equivalent. However, when $y$ embodies measurement error and/or when the forward model is imperfect, then there is no unique solution for $q$, so an objective function must be added in order to estimate $q$. LS approaches aim to minimize error objective functions like $g(q) = \sum_i (y_i - \hat{y}_i)^2$, where $i$ indexes the observed data points. Bayesian approaches contrastingly aim to maximize probability-based objective functions, which can be approximately interpreted as $h(q) = P(q|y)$, or maximizing the probability of observing the parameters $q$ given the data $y$. A more precise interpretation is that Bayesian approaches mutually maximize the maximum likelihoods of the parameters' posterior distributions. This procedure involves an iterative computational procedure in which distributions for all parameters are numerically generated based on assumed prior distributions, then updated based on the observed data, and the parameters' distributions are iteratively updated until all distributions' maxima are mutually maximized (Patil et al., 2010) (Appendix A & B).

When interpreted in the context of marker-based IK problems, $y$ represents the marker positions, $q$ represents the arbitrary kinematic parameters to be estimated, like joint angles or joint center locations, and $f(q)$ represents the forward kinematics model. LS approaches minimize predicted marker position errors $\sum_i (y_i - \hat{y}_i)^2$ and Bayesian approaches maximize the probability that a particular set of parameters $q$ could yield the observed marker positions $y$ (Appendix C).

To our knowledge Bayesian inference has not been previously applied to simple IK problems involving a small number of degrees of freedom. If Bayesian IK outperforms LS methods for simple IK problems, it is conceivable that it could be useful for more practical IK problems, including those involving STA. However the utility of Bayesian IK should first be assessed using simpler models in order to elucidate the model characteristics for which Bayesian IK and LS-IK diverge.

The purpose of this study was to compare Bayesian IK to LS-IK performance in estimates of planar kinematic chain poses and in planar rigid body rotations involving no STA. To this end we review the fundamentals of Bayesian inference (Appendix A & B), extend these concepts to basic IK problems (Appendix C), then demonstrate why the LS and Bayesian solutions can diverge, even for simple IK problems (Appendix D). The remainder of the main manuscript extends the simple IK discussed in Appendix C–D to slightly more complex planar mechanisms so that the corresponding Bayesian IK performance can be compared directly to the performances of two common STA-free LS techniques from the literature (Söderkvist and Wedin, 1993; Halvorsen et al., 1999).

## 2. Methods

Analyses were conducted in Python 3.6 (van Rossum, 2014) using Anaconda 3.5 (Anaconda, 2017). Appendices containing tutorials for Bayesian IK fundamentals, along with supplementary results, are available in this project's public repository at https://github.com/0todd0000/BayesIK.

### 2.1. Planar kinematic chain posture estimates

#### 2.1.1. Forward kinematics model

Kinematic chains were modeled using rigid segments which each contained one rigid marker plate with four markers (Fig. 1). Pilot simulations involving three, five and six markers were conducted but did not qualitatively affect results so are omitted in interest of space.

In order to permit direct comparison between least-squares (LS) and Bayesian techniques, local positions of the markers with respect to the proximal joint centers were assumed to be known. Forward kinematics (computing marker positions from mechanism pose) was implemented by setting the true global position of the proximal joint ($\boldsymbol{r}$) and all segment orientations ($\phi_i$), thereby yielding the global positions of all markers. From proximal to distal, three segment lengths were chosen to approximately reflect lower limb segment lengths: 0.45, 0.35 and 0.25 m.

Marker noise was modeled as multivariate Gaussian with uniform standard deviation $\sigma$. In Monte Carlo simulations $\sigma$ was drawn randomly from a uniform distribution with range: [0.1, 2.0 mm], representing the approximate range of marker measurement errors expected with modern motion capture systems (Chen et al., 1994; Windolf et al., 2008). Joint angles were drawn randomly from the uniform distribution spanning [0, 360 deg], thereby encompassing all possible mechanism postures.

For each simulation iteration the IK goal was to calculate mechanism posture ($\boldsymbol{r}$ and $\phi_i$) given the noisy marker positions, given the aforementioned forward kinematics model. A total of 1000 simulation iterations was conducted for each of the 1-, 2- and 3-link models depicted in Fig. 1. More simulation iterations did not qualitatively affect the results so are not reported.

#### 2.1.2. Least-squares approach

The least-squares (LS) approach minimized the function:

$$f(\tau) = \sum_i^M \left(\boldsymbol{r}_{mi} - \boldsymbol{r}'_{mi}(\tau)\right)^2 \tag{1}$$

where $\tau$ is the generalized posture vector (2), $i$ indexes the $M$ markers, and $\boldsymbol{r}_{mi}$ and $\boldsymbol{r}'_{mi}(\tau)$ are the 'measured' (simulated noisy) and calculated marker positions, respectively. For the 3-link chain mechanism the generalized posture vector is defined as:

$$\tau \equiv \{ r_x \quad r_y \quad \phi_1 \quad \phi_2 \quad \phi_3 \}^{\mathrm{T}} \tag{2}$$

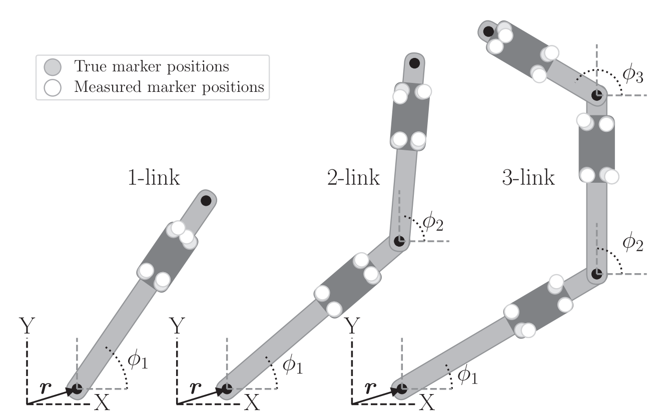


**Fig. 1.** Planar n-link kinematic chain models. The true local positions of all marker coordinates (i.e. positions with respect to each segment) were assumed to be known. The inverse kinematics goal was to estimate mechanism posture including: global position $\boldsymbol{r}$ and global orientations $\phi_i$ , where $i$ represents the $i$th joint.

The optimum solution for $\tau$ was found using quasi-Newton minimization (Nocedal and Wright, 2006) (p.136). The initial condition for numerical optimization was the true, known posture.

#### 2.1.3. Bayesian approach

Bayesian IK was implemented using PyMC (Patil et al., 2010). First a Bayesian model was constructed to represent the forward kinematics model (Fig. 2). Given the observed marker positions, the Bayesian solution was calculated using Markov-Chain Monte-Carlo simulations, with Metropolis stepping, which mutually maximized the maximum likelihoods of all posture parameters' posterior distributions. To avoid numerical optimization bias associated with the known true posture, Bayesian calculations were started from the LS solution, and with conservative, uniform priors. Specifically, the range for $r_x$ and $r_y$ was $\pm 100$ mm, and the range for $\phi_i$ was $\pm 180$ deg. The range for measurement precision parameter $\varepsilon$ (see Appendix C) was $[0.1\tau, 10\tau]$, where $\tau$ was the true measurement precision as defined by $\sigma$.

The LS and Bayesian IK solutions were compared both overall in terms of: median error, its interquartile range, and computational duration. They were also compared proportionately, by counting the number of (noisy) mechanism postures for which the Bayesian approach yielded lower error for each parameter. Bayesian calculations were performed over 100,000 iterations with a burn-in of 10,000 iterations and a thinning rate of 5. These iteration parameters were selected based on pilot simulations which were found to yield numerically stable results. Note that these 100,000 main Bayesian calculation iterations were performed separately for each of the aforementioned 1000 Monte Carlo iterations.

### 2.2. Planar rigid body rotation estimates

#### 2.2.1. Forward kinematics model

A single rigid body was rotated through angle $\Delta\phi$ about a proximal joint center located at position $\boldsymbol{r}$ (Fig. 3). During this rotation the $i$th marker was displaced according to displacement vector $\Delta\boldsymbol{r}_i$. The markers were assumed to be fixed to the rigid body but their true local positions were not assumed to be known.

#### 2.2.2. Least-squares approaches

Two LS approaches from the literature were adopted. The first (Söderkvist and Wedin, 1993) used singular value decomposition to solve the Procrustes problem of optimal transformation between the pre- and post- displacement frames. The second (Halvorsen et al., 1999) used a regular LS analysis of the displacement midpoints.

#### 2.2.3. Bayesian approach

The Bayesian IK approach used herein assumed that the observed pre-displacement marker positions $\boldsymbol{r}_i$ were the true positions, and thus that only the post-displacement marker positions $\boldsymbol{r}'_i$ were affected by the center of rotation $\boldsymbol{r}$ and angular displacement $\Delta\phi$ (Fig. 4). The IK goal was thus to mutually maximize the posterior maximum likelihoods of $\boldsymbol{r}$ and $\Delta\phi$ given the observed post-rotation marker positions $\boldsymbol{r}'_i$. Bayesian calculations were performed over 10,000 iterations with a burn-in of 5000 iterations and a thinning rate of 3. Like above, these iteration parameters were selected based on pilot simulations which were found to yield numerically stable results.

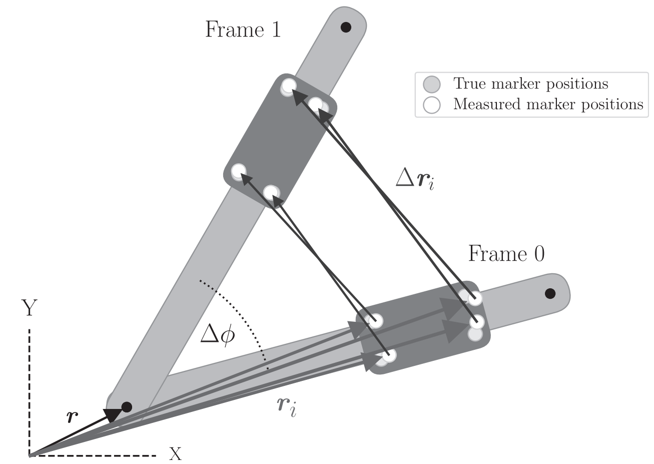


**Fig. 3.** Planar angular displacement-based inverse kinematics (IK) model. Based on initial marker positions $\boldsymbol{r}_i$ and marker displacements $\Delta\boldsymbol{r}_i$, the IK goal was to estimate the center of rotation $\boldsymbol{r}$ and the angular displacement $\Delta\phi$. Unlike the posture model (Fig. 1), and following least-squares techniques (Söderkvist and Wedin, 1993; Halvorsen et al., 1999), this model did not assume that true local marker positions were known.

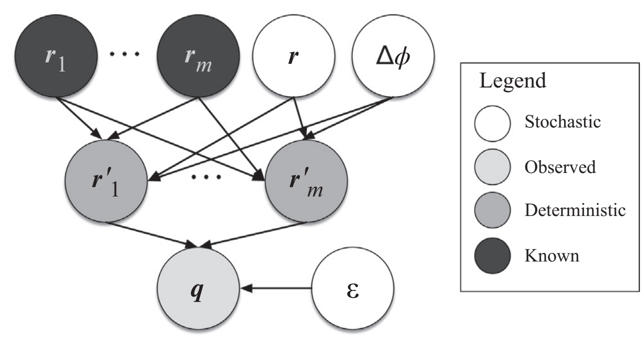


**Fig. 4.** Bayesian causal, forward kinematics model of planar rotation. In this model, initial marker positions $\boldsymbol{r}_i$ are assumed to have been measured accurately, and are thus treated as known constants. When values for the center of rotation $\boldsymbol{r}$ and angular displacement $\Delta\phi$ are decided, the true post-rotation marker positions $\boldsymbol{r}'_i$ are also known. These true positions plus measurement error $\epsilon$ yield the observed post-rotation marker positions $\boldsymbol{q}$. Like the posture model (Fig. 2), the Bayesian IK goal was to mutually maximize the posterior maximum likelihoods of all stochastic variables given the observations $\boldsymbol{q}$.

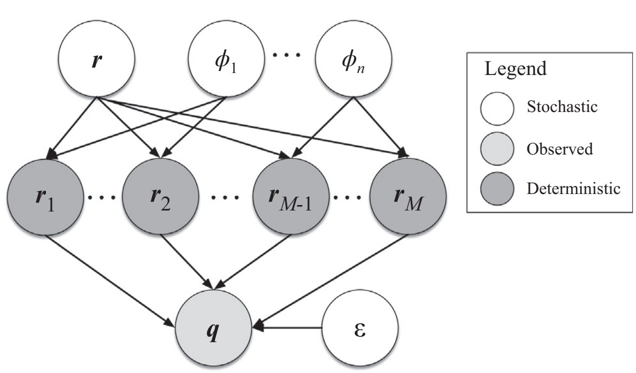


**Fig. 2.** Bayesian causal, forward kinematics model of the planar $n$-link kinematic chain postures depicted in Fig. 1. The top row represents the posture to be estimated: the global position of the first link's proximal joint center ($\boldsymbol{r}$) and segment orientations ($\phi_i$). Once values for these posture parameters are assigned, the true values of all global marker positions $\boldsymbol{r}_{Mj}$ are known. The observed marker positions $\boldsymbol{q}$ are a function of these true positions and random error $\varepsilon$. The Bayesian IK goal was to mutually maximize the posterior maximum likelihoods of all stochastic variables given the observations $\boldsymbol{q}$.

## 3. Results

### 3.1. Planar kinematic chain posture estimates

For the 1-link kinematic chain model (Fig. 1), Bayesian IK was found to yield smaller errors than LS-IK for all three postural parameters: $r_x$, $r_y$ and $\phi$ (Fig. 5a–c). In particular, while 81.0%, 81.1% and 98.4% of Bayesian IK errors for $r_x$, $r_y$ and $\phi$ were less than 0.1 mm, 0.1 mm and 0.1 deg, respectively, only 8.1%, 6.2% and 19.9% of the LS errors were smaller than these thresholds. Similar results were obtained for the 2- and 3-link models, but in interest of space are presented as supplementary material (Appendix E). Additionally, Bayesian IK yielded smaller error than LS-IK in more than 92% of all kinematic chain simulations (Table 1). Nevertheless, the computational resources required for the LS and Bayesian techniques were very different, with average (non-optimal) LS computational durations well under 0.5 s, and (non-optimal) Bayesian durations well over 2000 s (Table 2).

### 3.2. Planar rigid body rotation estimates

Bayesian IK tended to outperform the two LS methods (Figs. 5d–f and 6). In particular, 62.1%, 96.7% and 91.9% of Bayesian errors were less than 0.5 mm, 0.5 mm and 0.1 deg for $r_x$, $r_y$ and $\Delta\phi$, respectively, as compared with 30.7%, 18.1% and 16.1% for Söderkvist and Wedin (1993) and 3.6%, 9.3% and 6.0% for Halvorsen et al. (1999). Bayesian IK errors were smaller than LS errors in over 90% of all simulations, with the exception of $r_x$ for which Bayesian IK errors were lower than Söderkvist and Wedin (1993) in only 67.9% of simulations (Table 1).

Similar results were apparent when considering different noise amplitudes separately (Fig. 6): Bayesian IK errors tended to be much smaller than LS-IK errors for all except the smallest noise amplitudes, where the LS result tended to converge to Bayesian levels. Additionally, while LS errors tended to increase with marker noise (Fig. 6a–c), this trend was not apparent for the Bayesian errors (Fig. 6d–f).

Similar to the kinematic chain results, Bayesian IK computations required a much longer time than did the LS techniques. Whereas average computational durations for the LS techniques were on the order of 1 ms, Bayesian calculations required approximately 10 s (Table 2).

## 4. Discussion

This study summarized the fundamentals of Bayesian inference (Appendix A &B) as they apply to planar IK problems (Appendix C &D), and quantified the accuracies of Bayesian IK and LS-IK for a variety of planar, limb-like mechanisms. Results show that Bayesian IK and LS-IK solutions diverge even for simple 2-DOF planar mechanisms (Appendix D). For planar kinematic chain pose estimates and for planar rigid body rotations, Bayesian IK yielded order-of-magnitude more accurate IK estimates (Fig. 5), and these were more accurate than LS estimates in 95.3% of simulation results (Table 1). Bayesian IK was additionally less affected by marker noise amplitude than LS-IK (Fig. 6).

The current kinematic chain pose estimates used the identical forward kinematics model for both LS-IK and Bayesian IK. The LS-IK algorithm started from the true (known) IK solution, giving it the maximum chance to converge to the true solution. Contrastingly the Bayesian IK solution started from the LS-IK solution, giving it the maximum chance to converge to the LS-IK solution. Given this simulation framework, the fact that Bayesian IK was able to find a better solution than LS-IK provides strong evidence that Bayesian IK handles marker error in a numerically superior manner. This suggests more generally that probabilistic maximization as opposed to error minimization may be a better strategy for dealing with biomechanical error.

An additional advantage of Bayesian approaches not explored in this study is that it yields posterior distributions (Appendix A). These distributions represent explicit data-driven estimations of uncertainty, not only for marker noise, but also for all modeled kinematic parameters including joint angle and joint center. LS-IK contrastingly yields only implicit estimates of marker noise, as embodied in the objective function (Eq. 1). In other words, the LS-IK approach provides just a single scalar estimate of uncertainty (the objective function value) that represents only marker-level error. In contrast, Bayesian IK provides a comprehensive model of

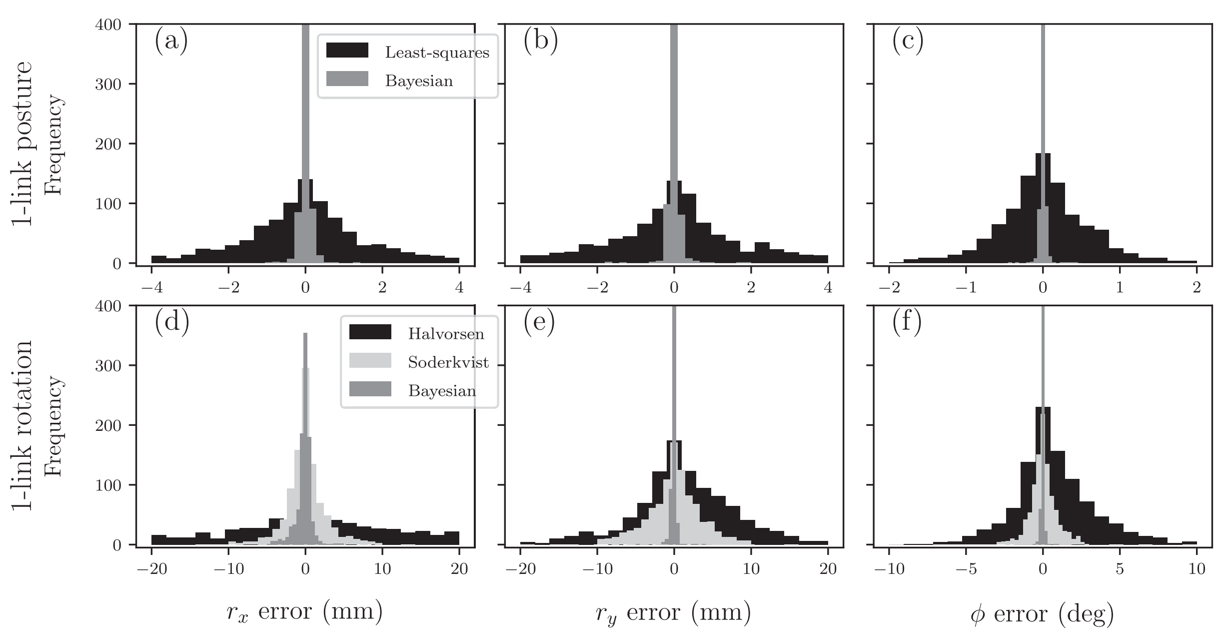


**Fig. 5.** Error distributions for estimates of 1-link posture (a–c) and rotation (d–f). The three columns represent the three estimated parameters (x position, y position, angular position/rotation). A total of 1000 simulations was conducted for each panel in which both marker noise amplitude and angular posture/ displacement were randomly varied. Similar results were obtained for 2-link and 3-link posture estimates (see Supplementary Material).

**Table 1**
Percentage of simulations in which Bayesian errors were smaller than least-squares errors (overall median percentage = 95.3%). Results are shown for the proximal joint center ($r_x$ and $r_y$) and rotation angles ($\phi$). For the rotation results (bottom two rows) $\phi_1$ represents angular displacement. Each percent value was derived from 1000 simulations.

| Model | Reference | $r_x$ | $r_y$ | $\phi_1$ | $\phi_2$ | $\phi_3$ |
|---|---|---|---|---|---|---|
| 1-link posture | – | 96.2 | 97.1 | 96.4 | – | – |
| 2-link posture | – | 94.7 | 94.2 | 95.3 | 94.8 | – |
| 3-link posture | – | 95.4 | 92.7 | 94.1 | 93.9 | 95.7 |
| 1-link rotation | Halvorsen et al. (1999) | 97.6 | 97.1 | 98.2 | – | – |
| 1-link rotation | Söderkvist et al. (1993) | 67.9 | 95.6 | 93.0 | – | – |

**Table 2**
Mean computational durations for the four IK methods. Durations for Bayesian IK are presented with units: seconds; all other durations have units: milliseconds.

| Model | Quasi-Newton (ms) | Halvorsen (ms) | Soderkvist (ms) | Bayesian (s) |
|---|---|---|---|---|
| 1-link posture | 78.8 | – | – | 2044.0 |
| 2-link posture | 306.2 | – | – | 4746.2 |
| 3-link posture | 285.9 | – | – | 2987.0 |
| 1-link rotation | – | 1.7 | 0.2 | 11.6 |

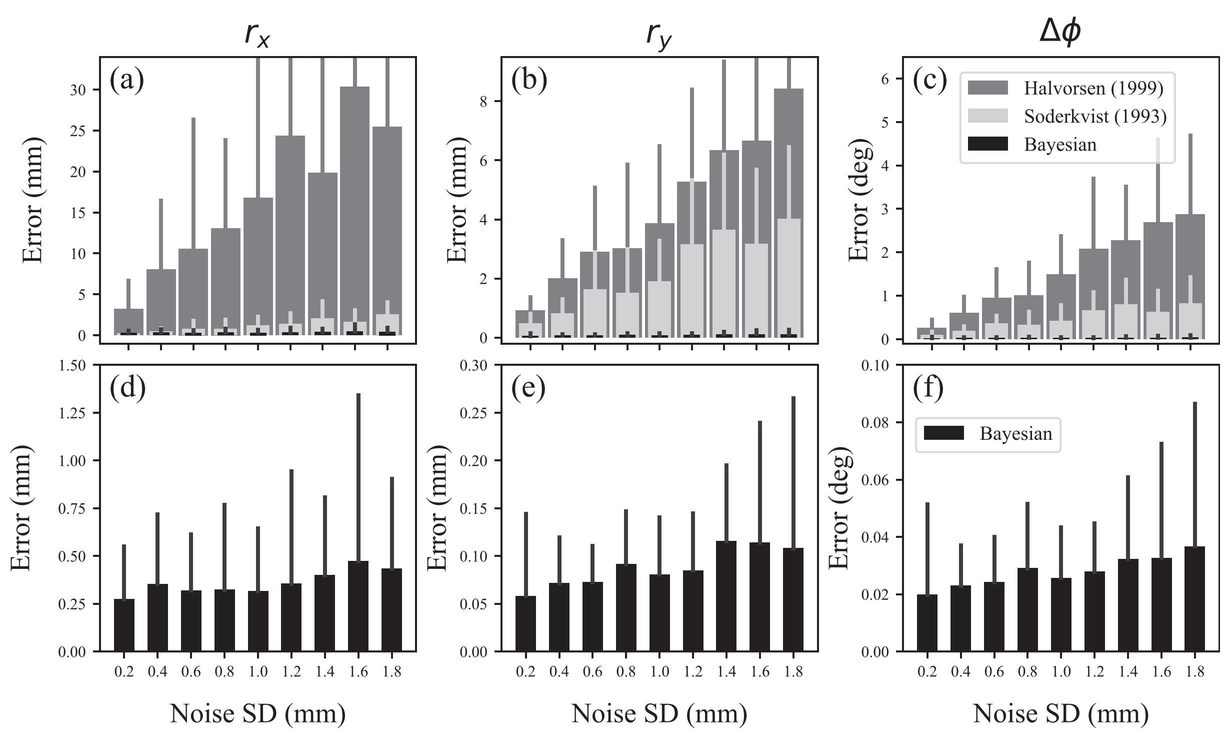


**Fig. 6.** IK errors vs. marker noise standard deviation (SD). Vertical bars indicate inter-quartile ranges. (a–c) Errors from all three methods. (d–f) Bayesian errors only (magnified for clarity).

uncertainty, in the form of posterior distributions for all modeled parameters, including (in this study): marker position, joint center, joint angle and marker measurement precision. Bayesian IK would also yield uncertainty models for additional parameters including: segment lengths, STA parameters, etc., if they were included in the forward kinematics model.

Bayesian IK's comprehensive model of kinematic uncertainty may be advantageous because the posterior distributions from IK can be propagated as prior distributions to subsequent data processing steps including smoothing and inverse dynamics. Ultimately this uncertainty may be important when attempting to make statistical inferences regarding the populations that the data are meant to represent. In other words, in all biomechanical analyses involving LS estimates, uncertainty is by definition considered only at inter-trial or inter-subject levels. Bayesian IK, and potentially also Bayesian inverse dynamics, could change this framework to also include uncertainty in each estimated parameter, including measured external forces and estimated joint and segment dynamics. This may be important because apparent inter-trial and/or inter-subject differences could conceivably fall within the range of Bayesian uncertainty, implying that these differences could be artifacts of computational uncertainty rather than functionally important differences. In contrast, if inter-trial and/or inter-subject differences fall well outside Bayesian IK's posterior distributions, it would reinforce current practice standards in which these differences are regarded as potentially functionally important.

Although LS-IK aims to minimize marker error, we show in Supplementary Material that Bayesian IK converges to LS-IK, unintuitively only when measurement error is presumed to be zero, or equivalently when measurements are presumed to be infinitely precise (Appendix C). For all other presumed marker error levels, Bayesian IK generally yields more accurate results than LS-IK (Appendix D). This paradox highlights the contrasting perspectives that LS-IK and Bayesian IK have of measured marker positions. LS-IK regards measured marker positions as known, and distributes presumed errors equally amongst the markers unless markers are non-uniformly weighted, in which case it would distribute errors equally in a weighted sense. Bayesian IK contrastingly regards measured marker positions more flexibly, as just one manifestation of an infinite continuum of measurement outcome possibilities. This flexible perspective allows Bayesian IK to ignore poor measurements in favor of the measurements which are more consistent with the forward stochastic model. Equivalently, Bayesian IK constitutes a form of objective marker weighting, in which markers are dynamically weighted based on the measured positions' probabilistic consistency with the forward kinematics model.

The main limitation of this study was that only planar poses and rotations were considered. For Bayesian IK to be useful for experimental biomechanics it would have to support estimates of 3D poses and transformations. Bayesian IK has already been implemented for such 3D problems in the robotics literature (Courty and Arnaud, 2008) so is likely also applicable to marker-based IK, but we leave this for future work.

A second limitation is that this study considered only single data frames: single postures or single displacements. It is conceivable that sequential Bayesian IK estimates over subsequent frames yield high frequency noise, and consequentially unrealistic accelerations. Nevertheless, it has been shown that augmentative IK algorithms like those including Kalman filtering, can reduce IK errors by on-the-order of 50% (De Groote et al., 2008), so any unrealistic frame-to-frame accelerations that emerge from Bayesian IK could likely by smoothed in a similar manner. On the other hand, it is conceivable that Bayesian IK itself could smooth multi-frame IK estimates, and even improve IK results for sequential frames if velocity and/or acceleration terms are incorporated into a multi-frame forward kinematics model. This type of model could be fitted to a sequence of measured marker positions, and this would be expected to improve the physical consistency of the estimated parameters relative to separate-frame fitting.

The main limitation of the current Bayesian IK implementation was computational time, which was order-of-magnitude larger than for LS-IK (Table 2). While the current Bayesian IK implementation is likely impractical for general laboratory use, it has been shown that optimized Bayesian IK can execute much more rapidly, and even faster than LS-IK for large degree-of-freedom problems (Courty and Arnaud, 2008). This computational efficiency may be useful for both real-time IK (Borbély et al., 2017) and real-time inverse dynamics applications (Pizzolato et al., 2016).

A final limitation was that we did not consider soft tissue artifact (STA). STA corrections could be easily incorporated within a Bayesian IK framework provided the STA can be modeled in a forward sense. However, since this was the first study of which we are aware to compare Bayesian IK to LS-IK for single joint kinematics, we chose to focus on simple-as-possible mechanisms to isolate effects of marker measurement error on IK error. In addition to a more complex forward model, STA-corrected Bayesian IK would involve both increased computational resource demands and experimental validation of the forward STA model. We therefore leave STA-corrected Bayesian IK for future work.

In summary, the current results suggest that Bayesian IK offers order-of-magnitude improved accuracy over LS-IK for estimations of both planar kinematic chain poses and planar rigid body rotations. This improved accuracy stems from differences in optimization objectives: whereas LS-IK minimizes error, Bayesian IK contrastingly maximizes probability. Additional studies are needed to elucidate the accuracy of Bayesian IK for 3D motion.

## Conflict of interest statement

The authors report no conflict of interest, financial or otherwise.

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## Supplementary material

Supplementary materials associated with this article can be found at https://github.com/0todd0000/BayesIK. PDF renderings of this supplementary material can be found in the online version of this article at https://doi.org/10.1016/j.jbiomech.2018.11.007.

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