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Wall shear stress in the Navier-Stokes equation: A commentary

In a recent article *(Xu et al, Computers in Biology and Medicine, 101(2018) p. 51-60)* the authors denoted Wall Shear Stress (WSS) as a vector field implying that such field has an intrinsic spatial direction of action. Xu et al. [1] tested different boundary conditions of transient CFD models of internal carotid artery (ICA) aneurysm. The purpose of their study was mainly to investigate the effect of aging, as represented by the blood flow waveform, on simulated hemodynamics of ICA aneurysm. In Figure 5 of their article, the authors showed what is called "WSS vector plot" where they correlated the so-called direction of WSS field to the oscillatory shear index (OSI). The authors imply that WSS is a vector-tensor field that has a direction of action. This is not quite uncommon in the interdisciplinary field of biomedical engineering [2,3] where the origins and derivation of Navier-Stokes equation (NSE) might not be very clear.

The Navier-Stokes equation in Cartesian vector notation can be written as, neglecting body forces:

$$\text{in x}-\text{direction}\quad \rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{1}$$

$$\text{in y}-\text{direction}\quad \rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{2}$$

$$\text{in z}-\text{direction}\quad \rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \tag{3}$$

The derivation of such equation in continuum mechanics dictates the use of infinitesimal differential element of the volume $\delta V = \delta x\delta y\delta z$ in non-accelerating frame of reference. Hence, the use of Cartesian differential operator $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ and Kronecker's delta δ_{ij} is valid. The Newtonian explanation for the relationship between shear stress and strain rate assumes that the former is linearly proportional to the latter. This has been known as the Newtonian fluid assumption. Based on which, the problem of expressing the stress tensor in equations (1)–(3) was solved by Navier and Stokes when they introduced the following substitution to replace the shear stress tensor with the shear rate (i.e. strain rate) tensor:

Normal stress components (scalar fields, similar to pressure) can be expressed as functions of the second viscosity coefficient $\lambda = -\frac{2}{3}\mu$, as hypothesized by Stokes [4,5], convective derivative $(\nabla\cdot V)$ and normal velocity gradient $\left(\frac{\partial u_i}{\partial x_i}\right)$, as:

in x − direction: $\tau_{xx} = \lambda(\nabla\cdot \mathrm{V}) + 2\mu\frac{\partial u}{\partial x}$

in y − direction: $\tau_{yy} = \lambda(\nabla\cdot V) + 2\mu\frac{\partial v}{\partial y}$

in z − direction: $\tau_{zz} = \lambda(\nabla\cdot V) + 2\mu\frac{\partial w}{\partial z}$,

The shear stress expressions, derived by Navier and Stokes, assumed symmetric tensor, thus equating each two reciprocating shear stress components, which was necessary to maintain the continuum and equilibrium assumptions of the equation, such that:

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{4}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{5}$$

$$\tau_{zy} = \tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{6}$$

Using Einstein summation convention [6], we write: $\tau_{ij} = \tau_{ji} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. The units of shear stress components in equations (4)–(6) are Pascal $(L^{-1}.\, M.\, T^{-2})$, which represent a scalar field with the space dimension (L) has negative power. In equations (4)–(6), the direction of action is represented by the subscript (j) while the plane of action is always normal to the first subscript (i). The meaning of equations (4)–(7) is simple; stress tensor symmetry. To instructively elucidate the physical meaning for the reader, equation (5) is considered. The shear stress component τ_{zx} and τ_{xz} represent the shear stress due to the force acting in x and z directions, on XY and YZ planes, respectively. Fig. 1 shows the analysis of such two forces on the infinitesimal fluid element moving with arbitrary velocity vector in Cartesian coordinates. In Fig. 1-a, the shear stress component τ_{ZX} acts on two faces namely a-b-f-e and d-c-g-h. The direction of such component on both faces is coincident with X-axis. In Fig. 1-b, the shear stress component τ_{XZ} acts on two faces namely b-c-g-f and a-d-h-e. The direction of such component on both faces is coincident with Z-axis. In equation (5), where $\tau_{ZX} = \tau_{XZ}$, the direction of shear stress in X and Z are equal, hence, the directionality of both tensor components is neutralized and represented only by one component. The physical meaning of equation (5), hence, is that the shear stress resulting values from two force components acting in two different directions, on two different planes, are equal. This applies to the remaining components of shear stress tensor. Therefore, the direction of action is neutralized in the symmetric assumption, and the shear stress on each plane has become a scalar field [7]. The physical meaning behind this neutralization of direction can be expressed in terms of strain rate isotropy. Needless to say that the latter assumption is only valid for fluids at equilibrium condition, which is the prime assumption upon which NSE was derived.

In computational hemodynamics, shear stress at the wall is evaluated based on the velocity gradients near to the wall, and the magnitude is evaluated as: $|\tau_w| = \mu\sqrt{\frac{1}{2}\left|S_{ij}\colon S_{ij}\right|_w}$ where $S_{ij} = \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$ is the strain rate tensor, and w denotes values in the first cell near to the wall because no-slip boundary conditions are always used at the wall. The so-called direction of WSS is often evaluated by decomposing the WSS field to tangential and binormal directions following the definition of unit normal and tangent vectors at every grid cell of the surface of interest [8]. The so-called *traction* field is introduced and the intrinsic

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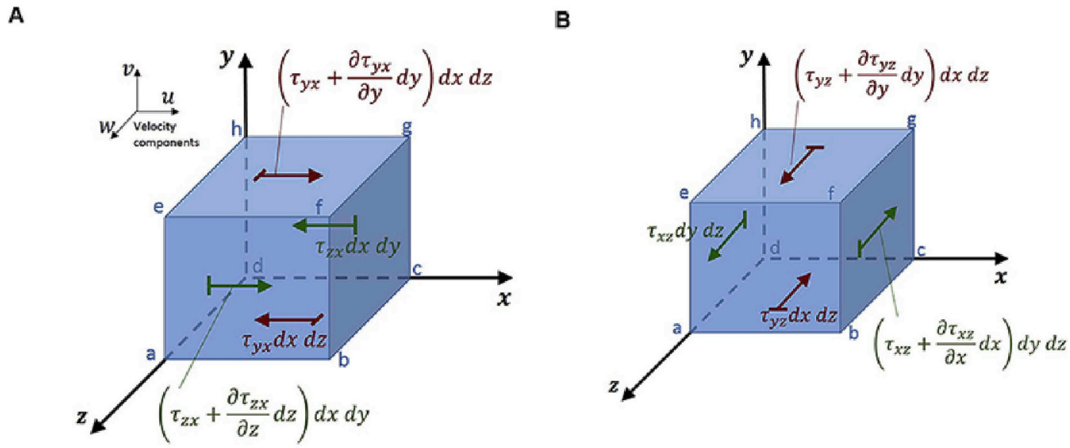

Fig. 1. Force analysis of a moving infinitesimal fluid element shows the symmetry of shear stress tensor. The infinitesimal element (defined by points a:h) has the dimensions $dx,\ dy\ and\ dz$ and the pressure and normal stress forces are not shown to simplify the figure. The shear stress components resulting from the viscous force in (a) X direction and (b) Z direction are shown.

derivatives are deduced to evaluate the magnitude of WSS on non-Euclidian surfaces, such as proposed by Cherubini et al. [9]. This means, however, that the WSS *obtained* directionality after the admission of the so-called *traction* field. Intrinsic derivatives enables the representation of WSS on complex anatomical geometries and decomposition to tangent and binormal components, this however does not imply in any sense that WSS is a vector in continuum Newtonian mechanics sense. It simply means that the intrinsic derivatives redefine the planes of action of WSS, not their direction of action. It is very important to always remember that a vector field must have intrinsic directionality, manifested in its units and dimensions [10]. The dimensions of so-called *WSS vector* remain $N.\ m^{-2} = kg.\ m^{-1}.\ s^{-2}$.

It can be argued that the shear stress tensor contains normal τ_{ii} and shear τ_{ij} components, hence, this can be considered as directionality if the unit-vectors ($\vec{n}$, $\vec{t}$) representing local orthogonal and parallel directions with respect to the wall are respectively considered. In other words, it can be argued that WSS is a vector *based on the geometry and morphology of the wall.* This is fundamentally erroneous, since the discrimination between scalar-tensors and vector-tensors must be made with respect to the frame of reference not the boundary conditions of the problem, such as wall morphology or orientation. One can do the same with any scalar (such as pressure), however, this does not make the pressure a *vector* field in any way. This is the essence of continuum mechanics and the very basic definition of differential analysis of fluid flow in non-inertial reference frames.

Nomenclature

p	Pressure (Pa, $L^{-1}MT^{-2}$)
t	Time (s, T)
$u,\ v,\ w$	Velocity in x, y and z directions ($m.\ s^{-1}$, LT^{-1})
τ_{ij}	Stress (Pa, $L^{-1}MT^{-2}$)
μ	First viscosity coefficient ($Pa.\ s$, $L^{-1}MT^{-1}$)
λ	Second viscosity coefficient ($Pa.\ s$, $L^{-1}MT^{-1}$)

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