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Comment

The Borsuk-Ulam theorem solves the curse of dimensionality: Comment on "The unreasonable effectiveness of small neural ensembles in high-dimensional brain" by Alexander N. Gorban et al.

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Gorban et al. [7] correlate multidimensional data spaces with high-dimensional nervous activities. This task requires huge computational power: indeed, the extraction of maximum value from the massive amount of gathered datasets entails newly-developed approaches for combining data-driven methods, physical modeling and algorithms capable of learning with limited, weak, or biased labels (Mars et al. [9]; Bergen et al. [3]).

On the other hand, when novel features are added in data mining, multiple measurements are combined and increases in data set dimensionality are achieved, the performance of the available classifiers tends to degrade (Barbour [1]). This excruciating issue is termed "the curse of dimensionality". When dimensions increase, the classifier's performance augments until the optimal number of features is reached. However, further increases in dimensionality result in performance decrease (Fig. 1A). In plain terms, the more the dimensions, the less the information available. This is a painstaking problem, if we want to think of the brain in terms of a multidimensional machine, as suggested by a growing amount of recent studies (see, for example, Bellmund et al. [2]; for a review, see Tozzi [16]).

The problem of the curse of dimensionality can be formulated in terms of geometry and topology too (Kůrková [8]). A n-sphere is a n-dimensional structure with positive (convex) curvature embedded in a $n+1$ Euclidean space, called a $n+1$-ball (Matousek [10]). For example, a circumference (1-dimensional sphere) surrounds a cd-rom (2-dimensional ball), while a thin plastic surface (2-dimensional sphere) surrounds a beach ball (3-dimensional ball). A 3-sphere (also called glome, or hypersphere) is a 3-dimensional elliptic surface, embedded in a 4-dimensional ball. What counts here is that the more the n-dimensions of the sphere, the less its volume, so that the hyperspheres' volumes tend to zero as their dimensionality tends to infinity (Fig. 1B). Indeed, in higher dimensions, the ball volume has a tendency to be concentrated near the equator (Fig. 1C). The volume of an n-sphere in terms of the volume of an $(n-2)$-ball of the same radius is:

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$$V_n(R) = \frac{2\pi R^2}{n} V_{n-2}(R)$$

Using two-dimension recursion formulas, it is easy to notice that the volume of an n-sphere of radius R approaches zero as n tends to infinity (Wang [17]). At each iteration, the new factor being multiplied into the volume is proportional to $1/n$, while the constant of proportionality $2\pi R^2$ is independent of n.

This counter-intuitive observation explains why the available information encompassed in a spherical volume decreases with increases in sphere's dimensions: as n-dimensionality augments, a larger percentage of the data tend to reside outside the equatorial feature space, which is the sole volume available for our observation. While the data encompassed in a few dimensions are packed and therefore easily detectable, the addition of further dimensions leads to "stretch" such data across that dimension, leaving them further apart. This causes the occurrence of extremely sparse information in higher dimensions, so that distance measures start losing their effectiveness to detect dissimilarity and become meaningless.

How to avoid the curse of dimensionality, when assessing multidimensional neural phase spaces? Apart from the canonical techniques used to achieve the "blessing of dimensionality" (Pereda et al. [11]), another, novel approach is available: the Borsuk-Ulam theorem (BUT), which has been widely used in physics, biology and neuroscience (for a review, see: Tozzi et al. [15]). The theorem states that (Borsuk [4]):

Every continuous map must identify a pair of antipodal points (*on* S^n).

In other words, if a sphere S^n is mapped continuously into a n-dimensional Euclidean space R^n, there is at least one pair of antipodal points on S^n mapping onto the same point of R^n. A pair of points are antipodal on a sphere, provided they are diametrically opposite each other. The poles of a sphere is an example. In the hands of K. Borsuk, BUT provides a building block in shape theory (Borsuk and Dydak [5]). In the recent years, the BUT's ingredients (antipodal points, n exponent and projections from lower to higher dimensions) have been modified to achieve a wide range of BUT variants. For an introduction, see Peters [12] and for a survey, see Tozzi et al. [15]. In these novel versions, the "points" are replaced by "regions", which stand for either mathematical, physical or biological features (Peters [12]). The term "matching description" means that the sets of regions display common feature values or symmetries. A symmetry break occurs when the symmetry is detectable at one level of observation, but is "hidden" at another level (Roldàn et al. [13]). Because BUT tells us that we can find, on an n-dimensional sphere, a pair of opposite points that have the same encoding on an $n-1$ sphere, this means that symmetries can be found when evaluating the system in higher dimensions, while they disappear (are hidden or broken) when we assess the same system in one dimension lower.

A BUT variant, termed **energy-BUT**, is particularly useful in our context (Tozzi and Peters [14]). A physical link does exist between the abstract concept of BUT and the energetic features of a system of two spheres S^n and S^{n-1}. We start from a manifold S^n equipped with a pair of antipodal features, standing for a symmetry. When these opposite features map to a n-Euclidean manifold (where S^{n-1} lies), a symmetry break/dimensionality reduction occurs, and a single feature is achieved. This means that the single mapping function on S^{n-1} displays energy parameters lower than the two corresponding antipodal functions on S^n (Fig. 1D). Therefore, decreases in dimensions give rise to energy decreases. In such a way, BUT yields physical quantities, because a system is achieved in which energetic changes occur among different dimensional levels. Energy-BUT concerns not just energy, but also information, providing a way to evaluate decreases in information in topological, other than thermodynamic, terms. Indeed, two antipodal features encompass more information than their single projection in a lower dimension.

Since the existence of one pair of mappings between different dimensions implies an overall change in thermodynamic and informational parameters, BUT is allowed to solve the curse of dimensionality. When a S^n sphere increases in dimensions, a single feature (say, a bit of information in S^2) becomes two features with matching description (say, two bits in S^3) (Fig. 1D). It is easy to see that the BUT-framed information amplifications occurring in high dimensions are concentrated near the equator, i.e., where the hypersphere's volume available for data assessment is located. Despite the expansion of dimensions shrinks the spherical volume leading to loss of information, the above-described BUT mechanism compensates and restores this failure, leading to an increase in information in higher-dimensional spheres.

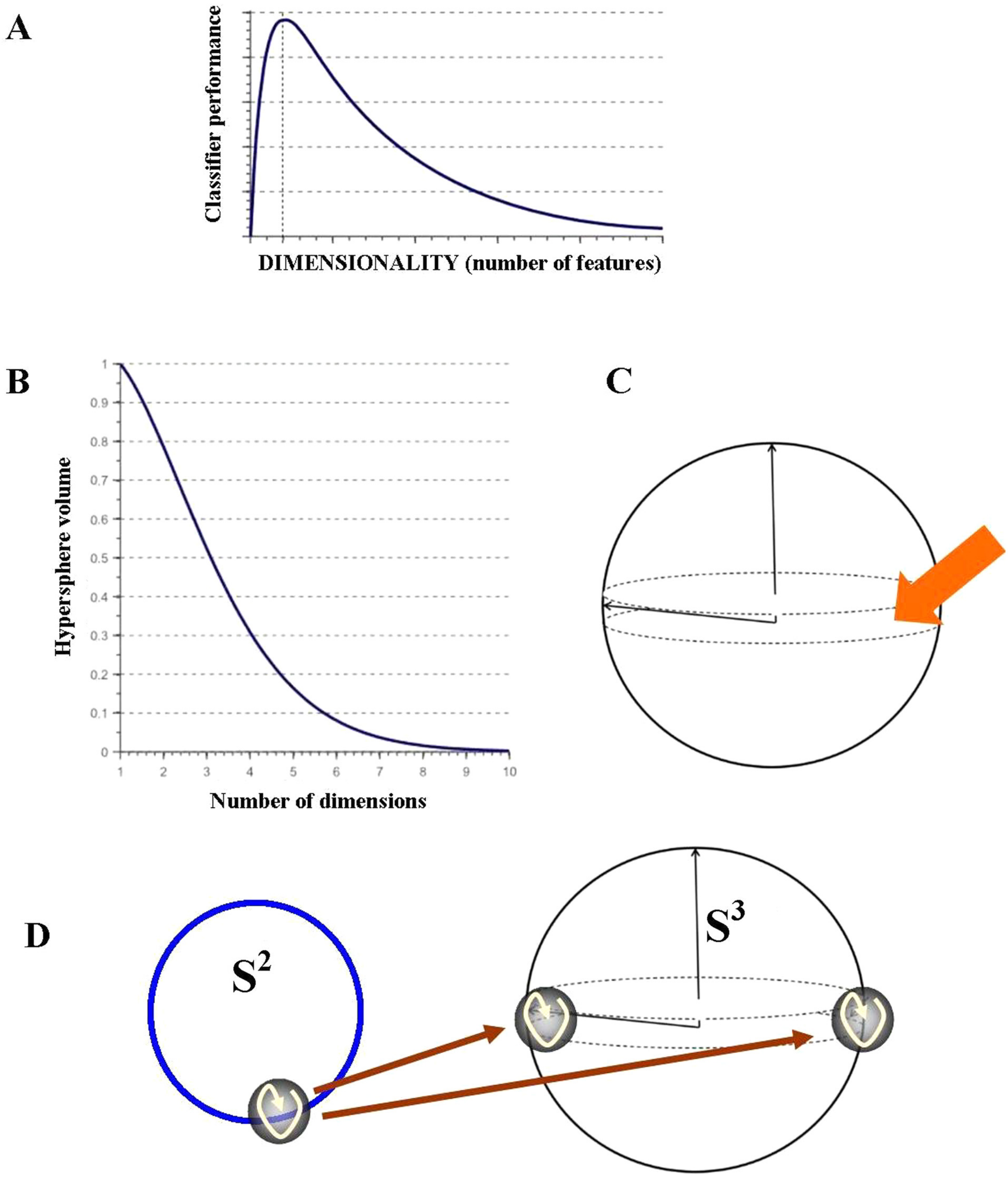

Fig. 1. How to solve the curse of dimensionality through the algebraic topological weapon of BUT. **Figure A**: an increase in number of measurements impairs classifier performance. **Figure B**: when assessing positive-curvature manifolds (spheres), an increase in dimensions leads to a decrease in volume. **Figure C**: Counterintuitively, the volume of high-dimensional balls is concentrated near their borders, in particular near their equator (red arrow). **Figure D**: in BUT terms, the manifold S^2 displays a single feature conformation (black oval containing a curved arrow), which becomes two features with matching description when mapped to a manifold S^3. It this case, the two antipodal features stand for two symmetric functions equipped with the same information content. In sum, a dimensional increase is correlated with an increase in available information, despite the spherical volume is decreased. **Figures A, B**: Modified from: http://www.visiondummy.com/2014/04/curse-dimensionality-affect-classification/; **Figure C**: modified from Gorban and Tyukin [6]; **Figure D**: modified from Tozzi et al. [15].

References

[1] Barbour DL. Precision medicine and the cursed dimensions. npj Digit Med 2019;2(4).
[2] Bellmund JLS, Gardenfors P, Moser EI, Doeller CF. Navigating cognition: spatial codes for human thinking. Science 2018;362(6415):eeat6766.
[3] Bergen KJ, Johnson PA, de Hoop MV, Beroza GC. Machine learning for data-driven discovery in solid Earth geosciences. Science 2019;363(6433):eaau0323. https://doi.org/10.1126/science.aau0323.
[4] Borsuk K. Dreisatzeuber die n-dimensionale euklidischesphare. Fundam Math 1933;XX:177–90.
[5] Borsuk K, Dydak J. What is the theory of shape?. Bull Aust Math Soc 1980;22:161–98.
[6] Gorban AN, Tyukin IY. Stochastic separation theorems. Neural Netw 2017;94:255–9. https://doi.org/10.1016/j.neunet.2017.07.014.
[7] Gorban AN, Makarov VA, Tyukin IY. The unreasonable effectiveness of small neural ensembles in high-dimensional brain. Phys Life Rev 2019;29:55–88. https://doi.org/10.1016/j.plrev.2018.09.005 [in this issue].
[8] Kůrková V. Some insights from high-dimensional spheres: Comment on "The unreasonable effectiveness of small neural ensembles in high-dimensional brain" by Alexander N. Gorban et al.. Phys Life Rev 2019. https://doi.org/10.1016/j.plrev.2019.03.014.
[9] Mars RB, Passingham RE, Jbabdi S. Connectivity fingerprints: from areal descriptions to abstract spaces. Trends Cogn Sci 2018 Sep 18;22(11):1026–37. https://doi.org/10.1016/j.tics.2018.08.009. pii:S1364-6613(18)30209-2.
[10] Matoušek J. Using the Borsuk–Ulam Theorem. Lectures on topological methods in combinatorics and geometry. Berlin-Heidelberg: Springer-Verlag; 2003.
[11] Pereda E, García-Torres M, Melián-Batista B, Mañas S, et al. The blessing of dimensionality: feature selection outperforms functional connectivity-based feature transformation to classify ADHD subjects from EEG patterns of phase synchronisation. PLoS ONE 2018;13(8):e0201660. https://doi.org/10.1371/journal.pone.0201660.
[12] Peters JF. Computational proximity. Excursions in the topology of digital images. In: Intelligent systems reference library. Berlin: Springer-Verlag; 2016.
[13] Roldán É, Martínez I, Parrondo JMR, Petrov D. Universal features in the energetics of symmetry breaking. Nat Phys 2014;10(6):457–61. http://doi.org/10.1038/nphys2940.
[14] Tozzi A, Peters JF. From abstract topology to real thermodynamic brain activity. Cogn Neurodyn 2017;11(3):283–92. https://doi.org/10.1007/s11571-017-9431-7.
[15] Tozzi A, Peters JF, Fingelkurts AA, Fingelkurts AA, Marijuán PC. Topodynamics of metastable brains. Phys Life Rev 2017;21:1–20. https://doi.org/10.1016/j.plrev.2017.03.001.
[16] Tozzi A. The multidimensional brain. Phys Life Rev 2019. https://doi.org/10.1016/j.plrev.2018.12.004.
[17] Wang X. Volumes of generalized unit balls. Math Mag 2005;8(5):390–5.